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RICCATI MATRIX DIFFERENTIAL EQUATIONS AND STURMIAN THEORY FOR LINEAR HAMILTONIAN SYSTEMS

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To my Teacher and Mentor

Contents

Preface

In this work we study the interrelations between two important concepts from the qualitative theory of differential equations. These are the Riccati differential equation and the separation or comparison properties of zeros of solutions of linear differential systems (called the Sturmian theory). These two concepts are connected by the well known mathematical object – the linear Hamiltonian differential system.

This work has been written for obtaining the academic qualification Associate Professor (docent) at the Masaryk University in Brno, Czech Republic. It contains research results achieved by the author in the years 2016–2021 and published in the papers

1. P. Šepitka, Genera of conjoined bases for (non)oscillatory linear Hamiltonian systems: extended theory, Journal of Dynamics and Differential Equations, 32 (2020), no. 3, 1139–1155,
2. P. Šepitka, Riccati equations for linear Hamiltonian systems without controllability condition, Discrete Continuous Dynamical Systems Series A, 39 (2019), no. 4, 1685–1730,
3. P. Šepitka, R. Šimon Hilscher, Comparative index and Sturmian theory for linear Hamiltonian systems, Journal of Differential Equations, 262 (2017), no. 2, 914–944,
4. P. Šepitka, R. Šimon Hilscher, Singular Sturmian separation theorems on unbounded intervals for linear Hamiltonian systems, Journal of Differential Equations, 266 (2019), no. 11, 7481–7524,
5. P. Šepitka, R. Šimon Hilscher, Singular Sturmian comparison theorems for linear Hamiltonian systems, Journal of Differential Equations, 269 (2020), no. 4, 2920–2955.

These publications are considered as the main sources, see items [**77**, **78**, **84**, **86**, **87**] in the list of references and Appendices A–E for more details. We also present some additional results in the context of the current literature, which are closely related to those in the above mentioned references, such as in papers [**34**, **85**, **88**, **89**] and in the monograph [**58**]. The habilitation thesis is considered as an extended commentary to the published results in the attached papers. We comment on our results in a broader context of the historical literature and the current development of the subject.

In Chapter 1 we discuss the Riccati matrix differential equations for possibly uncontrollable linear Hamiltonian systems. We show the variability of these Riccati matrix equations depending on the choice of a genus of conjoined bases, to which the considerations are restricted. The theory of genera of conjoined bases is presented as an introductory part. The results in Chapter 1 are based on the first two papers from the above list.

In Chapter 2 we present the Sturmian theory for linear Hamiltonian systems (i.e., the Sturmian separation and comparison theorems), which is based on the properties of the Riccati quotients – symmetric solutions of Riccati matrix differential equations. As a connecting tool we use the comparative index. This object was introduced by J. Elyseeva in 2007 in the connection with discrete oscillations. It was implemented into the continuous time theory by Elyseeva in 2016 and independently by the author and R. Šimon Hilscher in 2017 (in the third paper from the above list). By using the comparative index we are able to develop both regular and singular Sturmian theory, including the multiplicities of focal points at infinity.

Finally, we wish to mention that along with our investigations in the theory of possibly uncontrollable linear Hamiltonian systems we sometimes derive new results in other fields of mathematics, in particular in linear algebra (matrix analysis, theory of the Moore–Penrose pseudoinverses, and orthogonal projectors) or mathematical analysis (linear control systems). This is documented, for example, in the third section of the first paper mentioned above (Appendix A), in the last section of the fifth paper mentioned above (Appendix E), as well as in our previous papers [**79**, Appendix 1] and [**80**, Appendix 1]. We derived these results as needed tools for our investigations, but they may be of independent interest for other researchers.

I would like to express gratitude to my collaborator, colleague, friend, and former advisor Roman Šimon Hilscher for his continual support, willingness, fruitful discussions, and his advices and comments. Many thanks belong also to the heads of our scientific team, Zuzana Došlá and Petr Hasil.

Last, but not least, I thank my family, friends, and especially my Little Sun for their support.

Brno, May 2021 Peter Šepitka

CHAPTER 1

Riccati matrix differential equations

In this chapter we will present the theory of Riccati matrix differential equations associated to linear Hamiltonian differential systems. In particular, we will focus on our recent results on this subject, where we do not impose some traditional assumptions (as we explain below).

1.1. Introduction

Let $n \in \mathbb{N}$ be a given dimension, let $\mathcal{I} \subseteq \mathbb{R}$ be a given interval, and let $\mathcal{H} : \mathcal{I} \to \mathbb{R}^{2n\times 2n}$ be a given piecewise continuous matrix-valued function. Typically we will consider a compact interval $\mathcal{I} = [a, b]$ or an unbounded interval $\mathcal{I} = [a, \infty)$. As a main object of our study we consider the *linear Hamiltonian system*

$$y' = \mathcal{J}\mathcal{H}(t)\, y, \quad t \in \mathcal{I}, \tag{H}$$

where $\mathcal{J}$ is the canonical skew-symmetric $2n \times 2n$ matrix. In the $n \times n$ block notation we have

$$\mathcal{H}(t) = \begin{pmatrix} -C(t) & A^T(t) \\ A(t) & B(t) \end{pmatrix}, \quad t \in \mathcal{I}, \quad \mathcal{J} := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{1.1}$$

where $A, B, C : \mathcal{I} \to \mathbb{R}^{n\times n}$ are piecewise continuous functions such that $B(t)$ and $C(t)$ are symmetric. System (H) then has the equivalent form

$$x' = A(t)\, x + B(t)\, u, \quad u' = C(t)\, x - A^T(t)\, u, \quad t \in \mathcal{I}. \tag{1.2}$$

With system (H) we associate the *Riccati matrix differential equation*

$$Q' + Q\, A(t) + A^T(t)\, Q + Q\, B(t)\, Q - C(t) = 0, \quad t \in \mathcal{I}. \tag{R}$$

The connection of the Riccati equation (R) with system (H) is studied in many classical works see e.g. [21, 46, 58, 67–69]. It is known that under the Legendre condition

$$B(t) \geq 0 \quad \text{for all } t \in [a, \infty) \tag{1.3}$$

the Riccati equation (R) has many applications in various disciplines, such as in the oscillation and spectral theory [11, 21, 58, 67–69], filtering and prediction theory [57, 68], calculus of variations and optimal control theory [8, 12, 22, 43, 47, 49, 58, 63, 72, 73, 102–104], systems theory and control [55, 56], exponential dichotomy of perturbed linear Hamiltonian systems [43, 56, 64], and others (engineering, etc.). We recall that the Riccati matrix differential equation (R) has the distinguished property among the first order differential equations, namely it preserves the ordering of its solutions along the interval $[a, \infty)$, see [74, 75].

Classical theory of system (H) and equation (R) involves a *complete controllability* assumption, see e.g. [11, 21, 46, 58, 67, 69]. This assumption says that vector solutions (x, u) of system (1.1) are not degenerate on $\mathcal{I}$. Specifically, if the function x vanishes on a subinterval of $\mathcal{I}_0 \subseteq \mathcal{I}$, then also u vanishes on $\mathcal{I}_0$, and hence $(x, u) \equiv (0, 0)$ by the uniqueness of solutions. This condition is also known as the *identical normality* of system (H) on $\mathcal{I}$. A characterization of this condition in terms of focal points of conjoined bases of system (H) is presented in Proposition 2.1. When system (H) does not satisfy this complete controllability (identical normality) assumption or when this assumption is not imposed, then we say that system (H) is (possibly) *uncontrollable* or *abnormal*.

Let us recall several important results in this area. In [65], Reid showed that, under condition (1.3) and when system (H) is completely controllable and nonoscillatory, the Riccati equation (R) has the so-called distinguished solution $\hat{Q}(t)$ at infinity. More precisely, it is the smallest symmetric solution of (R) existing on an interval $[\alpha, \infty)$ for some $\alpha \geq a$. In the subsequent paper [66], Reid derived the existence and the minimality of the distinguished solution of (R) at infinity also for a noncontrollable system(H) by considering invertible principal solutions $(\hat{X}, \hat{U})$ of (H) at infinity.

1.2. Genera of conjoined bases

Recently, in [79–83] the author and Šimon Hilscher developed the theory of principal solutions at infinity and antiprincipal solutions at infinity (called also nonprincipal solutions at infinity in some literature) for a general nonoscillatory and possibly abnormal system (H). These notions will be recalled in Section 1.4. They showed the existence of principal and antiprincipal solutions at infinity, whose first component has the rank equal to any integer in the range between $n - d_\infty$ and n, where the number d_∞ is the maximal order of abnormality of (H), see below. The above general approach to principal and antiprincipal solutions at infinity naturally requires using the Moore–Penrose pseudoinverse [13, 19, 59], which in this context substitute the traditionally used invertible matrices. In the above references we also derived a classification of principal and antiprincipal solutions at infinity and their mutual limit properties at infinity. These results are based on the investigation of conjoined bases (X, U) of (H), which have eventually the same image of $X(t)$. The set of all such conjoined bases of (H) is called, according to [80, Definition 6.3], as

- a *genus of conjoined bases* of system (H),

and it is denoted by $\mathcal{G}$. This notion turned out to be a key tool for the study of analytic properties of conjoined bases of system (H), but also for the understanding of the algebraic structure of the set of all conjoined bases of (H). We showed that every genus $\mathcal{G}$ contains some principal solution of (H) at infinity, as well as some antiprincipal solution of (H) at infinity. Moreover, the orthogonal projector representing each genus $\mathcal{G}$ of conjoined bases satisfies a symmetric Riccati matrix differential equation. This result then allowed to obtain an exact description of the structure of the set of all genera of conjoined bases, in particular it forms a complete lattice. The minimal element of this lattice is the so-called *minimal genus* $\mathcal{G}_{\min}$, which contains all conjoined bases (X, U) with the eventual rank of $X(t)$ equal to smallest possible value $n - d_\infty$. On the other hand, the maximal element of this lattice is the so-called *maximal genus* $\mathcal{G}_{\max}$, which contains all conjoined bases (X, U) with the eventual rank of $X(t)$ equal to largest possible value n, i.e., with $X(t)$ eventually invertible. Note that in the completely controllable case we have $d_\infty = 0$, and hence the minimal and maximal genera coincide, i.e., there exists only one single genus $\mathcal{G} = \mathcal{G}_{\min} = \mathcal{G}_{\max}$. Therefore, in the study of abnormal systems (H) we obtain much wider structural variability and the theory of genera of conjoined bases provides a true guideline for potential development of the qualitative theory of these systems.

In [78] we extended the theory of genera of conjoined bases to arbitrary systems (H) by removing two key assumptions. Given the unbounded interval $\mathcal{I} = [a, \infty)$, we consider the case when

- the Legendre condition (1.3) is not assumed, and/or
- the system (H) may be oscillatory.

More precisely, in the new general definition of a genus $\mathcal{G}$ corresponding to a conjoined basis (X, U) of (H), see [78, Definition 4.3], we consider the subspace

$$\operatorname{Im} X(t) + \operatorname{Im} R_{\Lambda\infty}(t), \quad t \in [a, \infty), \tag{1.4}$$

where $R_{\Lambda\infty}(t)$ is the orthogonal projector onto the maximal subspace of eventually degenerate solutions $(x \equiv 0, u)$ of (H) at the point t. It is important to note that, according to [78, Theorem 4.7], every genus $\mathcal{G}$ can be represented by an orthogonal projector $R_{\mathcal{G}}(t)$ satisfying the Riccati type matrix differential equation

$$R_{\mathcal{G}}' - A(t)\, R_{\mathcal{G}} - R_{\mathcal{G}}\, A^T(t) + R_{\mathcal{G}}\, [A(t) + A^T(t)]\, R_{\mathcal{G}} = 0, \quad t \in [a, \infty). \tag{1.5}$$

Also in this much more general case it is possible to show that the set of all genera of conjoined bases of system (H) forms a complete lattice, see [78, Theorem 4.14].

In general, following the standard notation used in [66, Section 3] and [82, Section 2], for a given $\alpha \in [a, \infty)$ we denote by $\Lambda[\alpha, \infty)$ the linear space of n-dimensional piecewise continuously differentiable vector-valued functions u which correspond to the solutions $(x \equiv 0, u)$ of system (H) on $[\alpha, \infty)$. The space $\Lambda[\alpha, \infty)$ is finite-dimensional with $d[\alpha, \infty) := \dim \Lambda[\alpha, \infty) \leq n$. The number $d[\alpha, \infty)$ is called the *order of abnormality* of system (H) on the interval $[\alpha,\infty)$. According to [80, Section 6] there exists the limit

$$d_\infty := \lim_{t\to\infty} d[t,\infty) = \max_{t\in[a,\infty)} d[t,\infty), \quad 0 \le d_\infty \le n, \tag{1.6}$$

which we call the *maximal order of abnormality* of (H). Moreover, we define the point

$$\alpha_\infty := \min\{\alpha \in [a,\infty), \quad d[\alpha,\infty) = d_\infty\}. \tag{1.7}$$

Following [78] we consider the orthogonal projector

$$R_{\Lambda\infty}(t) := \mathcal{P}_{\mathcal{W}_t^\perp}, \quad \text{where} \quad \mathcal{W}_t := \Lambda_t[\alpha_\infty,\infty), \quad t \in [\alpha_\infty,\infty). \tag{1.8}$$

Here the set $\Lambda_t[\alpha_\infty,\infty)$ is the subspace in $\mathbb{R}^n$ of the values $u(t)$ of functions $u \in \Lambda[\alpha_\infty,\infty)$ at the point $t \in [\alpha_\infty,\infty)$. The orthogonal projector $R_{\Lambda\infty}(t)$ plays a crucial role in the new theory of genera of conjoined bases of system (H). In the remaining part of this section we present the main results from [78], see also Appendix A.

Definition 1.1 (Genus of conjoined bases)**.** Let (X_1,U_1) and (X_2,U_2) be two conjoined bases of (H). We say that (X_1,U_1) and (X_2,U_2) *have the same genus* (or they *belong to the same genus*) if there exists $\alpha \in [\alpha_\infty,\infty)$ such that

$$\operatorname{Im} X_1(t) + \operatorname{Im} R_{\Lambda\infty}(t) = \operatorname{Im} X_2(t) + \operatorname{Im} R_{\Lambda\infty}(t), \quad t \in [\alpha,\infty),$$

where α_∞ is defined in (1.7).

From Definition 1.1 it follows that there exists a partition of the set of all conjoined bases of (H) into disjoint classes of conjoined bases with the same genus. We will interpret each class $\mathcal{G}$ as a genus itself. The following result provides a fundamental property of conjoined bases of (H) with the same genus.

Theorem 1.2. *Let (X_1,U_1) and (X_2,U_2) be conjoined bases of (H). Then the following statements are equivalent.*

(i) *The conjoined bases (X_1,U_1) and (X_2,U_2) belong to the same genus $\mathcal{G}$.*
(ii) *The equality $\operatorname{Im} X_1(t) + \operatorname{Im} R_{\Lambda\infty}(t) = \operatorname{Im} X_2(t) + \operatorname{Im} R_{\Lambda\infty}(t)$ holds for every $t \in [\alpha_\infty,\infty)$.*
(iii) *The equality $\operatorname{Im} X_1(t) + \operatorname{Im} R_{\Lambda\infty}(t) = \operatorname{Im} X_2(t) + \operatorname{Im} R_{\Lambda\infty}(t)$ holds for some $t \in [\alpha_\infty,\infty)$.*

Given a genus $\mathcal{G}$ of conjoined bases of (H), the subspace $\operatorname{Im} X(t) + \operatorname{Im} R_{\Lambda\infty}(t)$ does not depend on the particular choice of such a conjoined basis (X,U) belonging to $\mathcal{G}$. Therefore, the orthogonal projector onto $\operatorname{Im} X(t) + \operatorname{Im} R_{\Lambda\infty}(t)$, i.e., the matrix

$$R_{\mathcal{G}}(t) := \mathcal{P}_{\mathcal{V}_t}, \quad \text{where} \quad \mathcal{V}_t := \operatorname{Im} X(t) + \operatorname{Im} R_{\Lambda\infty}(t), \quad t \in [\alpha_\infty,\infty), \tag{1.9}$$

is uniquely determined for each genus $\mathcal{G}$. The following two theorems provide basic properties of orthogonal projectors $R_{\mathcal{G}}(t)$ defined in (1.9). Moreover, they show how to classify a genus $\mathcal{G}$ of conjoined bases of (H) via its associated projector $R_{\mathcal{G}}(t)$ in (1.9).

Theorem 1.3. *Let $\mathcal{G}$ be a genus of conjoined bases of (H) and let $R_{\mathcal{G}}(t)$ be the orthogonal projector defined in (1.9). Then the matrix $R_{\mathcal{G}}(t)$ is a solution of the Riccati equation (1.5) on $[\alpha_\infty,\infty)$ and the inclusion $\operatorname{Im} R_{\Lambda\infty}(t) \subseteq \operatorname{Im} R_{\mathcal{G}}(t)$ holds for every $t \in [\alpha_\infty,\infty)$.*

Theorem 1.4. *Let $\alpha \in [\alpha_\infty,\infty)$ be fixed and let $R \in \mathbb{R}^{n\times n}$ be an orthogonal projector satisfying $\operatorname{Im} R_{\Lambda\infty}(\alpha) \subseteq \operatorname{Im} R$. Then there exists a unique genus $\mathcal{G}$ of conjoined bases of (H) such that its corresponding orthogonal projector $R_{\mathcal{G}}(t)$ in (1.9) satisfies $R_{\mathcal{G}}(\alpha) = R$.*

We note that for every genus $\mathcal{G}$ its associated orthogonal projector $R_{\mathcal{G}}(t)$ in (1.9), as a solution of (1.5) on $[\alpha_\infty,\infty)$, has constant rank on the whole interval $[\alpha_\infty,\infty)$, i.e.,

$$r_{\mathcal{G}} := \operatorname{rank} R_{\mathcal{G}}(t), \quad t \in [\alpha_\infty,\infty). \tag{1.10}$$

In this context, we may adopt for the number $r_{\mathcal{G}}$ the terminology *rank of the genus* $\mathcal{G}$ and write $\operatorname{rank}\mathcal{G} := r_{\mathcal{G}}$, compare also with [81, Remark 6.4]. In particular, we have

$$n - d_\infty \le \operatorname{rank}\mathcal{G} \le n. \tag{1.11}$$

Let us denote by the symbol Γ the set of all genera of conjoined bases of (H). In the next definition we introduce an ordering on the set of all genera of conjoined bases of (H) in terms of their corresponding orthogonal projectors in (1.9).

Definition 1.5. Let $\mathcal{G}$ and $\mathcal{H}$ be two genera of conjoined bases of (H) and let $R_{\mathcal{G}}(t)$ and $R_{\mathcal{H}}(t)$ be their corresponding orthogonal projectors in (1.9), respectively. We say that the genus $\mathcal{G}$ is *below* the genus $\mathcal{H}$ (or that the genus $\mathcal{H}$ is *above* the genus $\mathcal{G}$) and we write $\mathcal{G} \preceq \mathcal{H}$ if the inclusion $\operatorname{Im} R_{\mathcal{G}}(t) \subseteq \operatorname{Im} R_{\mathcal{H}}(t)$ holds for all $t \in [\alpha_\infty, \infty)$.

Theorem 1.6. *The relation $\preceq$ from Definition 1.5 is an ordering on the set Γ.*

We note that the genera $\mathcal{G}$ and $\mathcal{H}$ satisfy $\mathcal{G} \preceq \mathcal{H}$ if and only if the inclusion $\operatorname{Im} R_{\mathcal{G}}(\alpha) \subseteq \operatorname{Im} R_{\mathcal{H}}(\alpha)$ holds for some $\alpha \in [\alpha_\infty, \infty)$. In particular, if the orthogonal projector $R_{\mathcal{G}}(t)$ satisfies $R_{\mathcal{G}}(t) = R_{\Lambda\infty}(t)$ on $[\alpha_\infty, \infty)$, then the genus $\mathcal{G} = \mathcal{G}_{\min}$ is called *minimal*, while if $R_{\mathcal{G}}(t) \equiv I$ on $[\alpha_\infty, \infty)$, then the genus $\mathcal{G} = \mathcal{G}_{\max}$ is called *maximal*.

Theorem 1.7. *The ordered set $(\Gamma, \preceq)$ is a complete lattice. In particular, the minimal genus $\mathcal{G}_{\min}$ is the least element of Γ with respect to the ordering $\preceq$, while the maximal genus $\mathcal{G}_{\max}$ is the greatest element of Γ with respect to $\preceq$.*

We can describe explicitly the infimum $\mathcal{G} \wedge \mathcal{H}$ and the supremum $\mathcal{G} \vee \mathcal{H}$ of two genera $\mathcal{G}$ and $\mathcal{H}$ of the set Γ. More precisely, if $R_{\mathcal{G}}(t)$ and $R_{\mathcal{H}}(t)$ are the orthogonal projectors associated to the genera $\mathcal{G}$ and $\mathcal{H}$, then $\mathcal{G} \wedge \mathcal{H}$ is the genus of conjoined bases corresponding to the orthogonal projector onto the subspace $\operatorname{Im} R_{\mathcal{G}}(t) \cap \operatorname{Im} R_{\mathcal{H}}(t)$ on $[\alpha_\infty, \infty)$, and $\mathcal{G} \vee \mathcal{H}$ is the genus of conjoined bases corresponding to the orthogonal projector onto the subspace $\operatorname{Im} R_{\mathcal{G}}(t) + \operatorname{Im} R_{\mathcal{H}}(t)$ on $[\alpha_\infty, \infty)$.

The next theorem characterizes the conjoined bases of (H) belonging to the minimal genus $\mathcal{G}_{\min}$. We also show that the principal solution of (H) at the point $\alpha \in [\alpha_\infty, \infty)$ belongs to the minimal genus $\mathcal{G}_{\min}$. The principal solution of (H) at the point α is defined as the matrix solution of (H) satisfying the initial condition

$$X_\alpha(\alpha) = 0, \quad U_\alpha(\alpha) = I, \tag{1.12}$$

Theorem 1.8. *Let (X, U) be a conjoined basis of (H). Then (X, U) belongs to the minimal genus $\mathcal{G}_{\min}$ if and only if the inclusion $\operatorname{Im} X(t) \subseteq \operatorname{Im} R_{\Lambda\infty}(t)$ holds for some (and hence for every) $t \in [\alpha_\infty, \infty)$. In particular, for every $\alpha \geq \alpha_\infty$ the principal solution (X_α, U_α) at the point α belongs to $\mathcal{G}_{\min}$.*

In the final theorem of this section we provide important properties of nonoscillatory conjoined bases from a given genus $\mathcal{G}$.

Theorem 1.9. *Let $\mathcal{G}$ be a genus of conjoined basis of (H) with the corresponding orthogonal projector $R_{\mathcal{G}}(t)$ in (1.9). Moreover, let (X, U) be a conjoined basis of (H) with constant kernel on $[\alpha, \infty) \subseteq [\alpha_\infty, \infty)$ such that (X, U) belongs to $\mathcal{G}$ and let $R(t)$ be the orthogonal projector onto the subspace $\operatorname{Im} X(t)$ for every $t \in [\alpha, \infty)$. Then the equality $R_{\mathcal{G}}(t) = R(t)$ holds for all $t \in [\alpha, \infty)$.*

In the final paragraph of this section we comment on the connection of the above results with those in [80], where it is assumed that the Legendre condition (1.3) holds and that system (H) is nonoscillatory. In particular, for every conjoined basis (X, U) of (H) there exists $\alpha \in [\alpha_\infty, \infty)$ such that (X, U) has constant kernel on $[\alpha, \infty)$. Moreover, let $\mathcal{G}$ be the genus of conjoined bases such that $(X, U) \in \mathcal{G}$ and let $R_{\mathcal{G}}(t)$ be its corresponding orthogonal projector in (1.9). Then the rank $r_{\mathcal{G}}$ of $\mathcal{G}$ defined in (1.10) coincides with the rank r of any conjoined basis (X, U) of the genus $\mathcal{G}$. In view of (1.11) we then obtain that

$$n - d_\infty \leq \operatorname{rank} X(t) \leq n, \quad t \in [\alpha, \infty), \tag{1.13}$$

for every conjoined basis (X, U) of system (H) with constant kernel on the interval $[\alpha, \infty)$. Moreover, by Theorem 1.9 we have that $\operatorname{Im} X(t) = \operatorname{Im} R_{\mathcal{G}}(t)$ on $[\alpha, \infty)$. Therefore, from (1.9) and Definition 1.1 it follows that two conjoined bases (X_1, U_1) and (X_2, U_2) of (H) belong to the same genus if and only if there exists $\alpha \in [\alpha_\infty, \infty)$ such that the equality $\operatorname{Im} X_1(t) = \operatorname{Im} X_2(t)$ hold for every $t \in [\alpha, \infty)$. This observation shows that the concept given in Definition 1.1 generalizes the definition of genus of conjoined bases introduced in [80, Definition 6.3] for a nonoscillatory system (H). We also note that the result about the structure of the set of all genera of conjoined bases presented in Theorems 1.6 and 1.7 are in full agreement with the corresponding results in [82, Section 4]. Finally, the result in Theorem 1.8 generalizes [82, Proposition 4.7] to a possibly oscillatory system (H).

1.3. Riccati matrix differential equations

The study of Riccati matrix differential equations associated with uncontrollable linear Hamiltonian systems is also motivated by several situations in the literature. For example, in [73, pg. 886], [8, pp. 621–622], [48, Sections 4 and 6], and [49, pp. 17–18] the authors use a cascade system of three differential equations for the investigation of calculus of variations or optimal control problems with variable endpoints – the Riccati equation (R), a linear differential equation, and an integrator. These three differential equations are together equivalent to a Riccati equation in dimension $2n$, which corresponds to an uncontrollable system (H) in dimension $4n$. This connection is discussed in details in [48, Remark 6.3]. Among other situations we also mention the occurrence of the symmetric solutions of the implicit Riccati equation

$$R_{\mathcal{G}}(t)\,[Q' + Q\,A(t) + A^T(t)\,Q + Q\,B(t)\,Q - C(t)]\,R_{\mathcal{G}}(t) = 0, \quad t \in \mathcal{I}, \tag{1.14}$$

in the study of nonnegative quadratic functional associated with possibly an uncontrollable system (H), see [50, Section 6].

The above mentioned extended theory of genera of conjoined bases of system (H) based on the subspaces $\operatorname{Im} R_{\mathcal{G}}(t)$ in (1.9) points to new possibilities how to deal with the Riccati matrix differential equations in the context of abnormal linear Hamiltonian systems (H). In particular, it shows how to implement in a proper way the theory of the Riccati type differential equations (R) or (1.14) into the theory of linear Hamiltonian systems (H). Consider the unbounded interval $\mathcal{I} = [a, \infty)$. The presented approach is novel in three aspects. Namely,

- we do not require any controllability assumption on system (H),
- for every genus $\mathcal{G}$ we associate a Riccati equation

$$Q' + Q\,\mathcal{A}(t) + \mathcal{A}^T(t)\,Q + Q\,\mathcal{B}(t)\,Q - \mathcal{C}(t) = 0, \quad t \in [\alpha_\infty, \infty), \tag{$\mathcal{R}$}$$

  where the coefficients $\mathcal{A}(t)$, $\mathcal{B}(t)$, and $\mathcal{C}(t)$ are given by

$$\left.\begin{aligned} \mathcal{A}(t) &:= A(t)R_{\mathcal{G}}(t) - A^T(t)\,[I - R_{\mathcal{G}}(t)], \\ \mathcal{B}(t) &:= B(t), \\ \mathcal{C}(t) &:= R_{\mathcal{G}}(t)\,C(t)\,R_{\mathcal{G}}(t), \end{aligned}\right\} \quad t \in [\alpha_\infty, \infty), \tag{1.15}$$

  with the corresponding orthogonal projector $R_{\mathcal{G}}(t)$ defined in (1.9),
- we show that every such a Riccati equation ($\mathcal{R}$) possesses a distinguished solution at infinity (defined in a suitable way), which corresponds to a principal solution of (H) at infinity from the genus $\mathcal{G}$.

Given a genus $\mathcal{G}$ of conjoined bases of (H), we show a fundamental connection between the symmetric solutions $Q(t)$ of ($\mathcal{R}$) on $[\alpha, \infty)$ with some $\alpha \geq \alpha_\infty$ satisfying

$$\operatorname{Im} Q(t) \subseteq \operatorname{Im} R_{\mathcal{G}}(t), \quad t \in [\alpha, \infty), \tag{1.16}$$

and the conjoined bases (X, U) of (H) with constant kernel on $[\alpha, \infty)$, which belong to $\mathcal{G}$. This allows us to define in a proper way a distinguished solution $\hat{Q}(t)$ at infinity for each Riccati equation ($\mathcal{R}$), which corresponds to a principal solution $(\hat{X}, \hat{U})$ of (H) at infinity in $\mathcal{G}$. Then for every symmetric solution $Q(t)$ of ($\mathcal{R}$) on $[\alpha, \infty)$ with (1.16) there exists a distinguished solution $\hat{Q}(t)$ of ($\mathcal{R}$) satisfying the inequality

$$Q(t) \geq \hat{Q}(t) \quad \text{on } [\alpha, \infty). \tag{1.17}$$

The above results are particularly important for the minimal genus $\mathcal{G} = \mathcal{G}_{\min}$, which is formed by the conjoined bases (X, U) of (H) with minimal possible rank of the matrix $X(t)$, i.e., with $\operatorname{rank} X(t) = n - d_\infty$ on $[\alpha, \infty)$. In this case the associated distinguished solution $\hat{Q}_{\min}(t)$ at infinity is unique and minimal among all symmetric solutions $Q(t)$ of ($\mathcal{R}$) satisfying (1.16). This latter situation generalizes the classical controllable results of Reid and Coppel [21, 65, 67], since in this case $d_\infty = 0$ and the orthogonal projector $R_{\mathcal{G}}(t) \equiv I$ on $[a, \infty)$, so that the Riccati equation ($\mathcal{R}$) reduces to (R). We note that the original results by Reid [66, 68] for noncontrollable system (H) and Riccati equation (R) correspond in our new theory to the maximal genus $\mathcal{G} = \mathcal{G}_{\max}$ of conjoined bases (X, U) with eventually invertible matrix $X(t)$, i.e., to $R_{\mathcal{G}}(t) \equiv I$ on $[a, \infty)$. Therefore, the present study can be regarded as a generalization and completion of the theory of the Riccati equations (R) for completely controllable systems (H) using the minimal genus $\mathcal{G} = \mathcal{G}_{\min}$, as well as the noncontrollable systems (H) using the maximal genus $\mathcal{G} = \mathcal{G}_{\max}$. In the remaining part of this section we present the main results from [77, Sections 4–6], see also Appendix B.

In the first set of results (Theorems 1.10–1.13) we describe basic properties of solutions of the Riccati equation ($\mathcal{R}$).

Theorem 1.10. *Let $\mathcal{G}$ be a genus of conjoined bases of (H) with the corresponding orthogonal projector $R_{\mathcal{G}}(t)$ in (1.9) and let $Q(t)$ be a solution of the Riccati equation ($\mathcal{R}$) on $[\alpha, \infty) \subseteq [\alpha_\infty, \infty)$ with α_∞ in (1.7). Then also the matrices $R_{\mathcal{G}}(t)\, Q(t)$, $Q(t)\, R_{\mathcal{G}}(t)$, and $R_{\mathcal{G}}(t)\, Q(t)\, R_{\mathcal{G}}(t)$ solve ($\mathcal{R}$) on $[\alpha, \infty)$.*

Theorem 1.11. *With the assumptions and notations of Theorem 1.10, the matrix $Q(t)$ satisfies the inclusion $\operatorname{Im} Q(t) \subseteq \operatorname{Im} R_{\mathcal{G}}(t)$, resp. the inclusion $\operatorname{Im} Q^T(t) \subseteq \operatorname{Im} R_{\mathcal{G}}(t)$, for all $t \in [\alpha, \infty)$ if and only if the inclusion $\operatorname{Im} Q(t_0) \subseteq \operatorname{Im} R_{\mathcal{G}}(t_0)$, resp. the inclusion $\operatorname{Im} Q^T(t_0) \subseteq \operatorname{Im} R_{\mathcal{G}}(t_0)$, holds for some point $t_0 \in [\alpha, \infty)$.*

Theorem 1.12. *Let $\mathcal{G}$ be a genus of conjoined bases of (H) with the corresponding matrix $R_{\mathcal{G}}(t)$ in (1.9) and let $Q(t)$ and $\tilde{Q}(t)$ be symmetric solutions of the Riccati equation ($\mathcal{R}$) on $[\alpha, \infty) \subseteq [\alpha_\infty, \infty)$. Then the quantities*

$$\operatorname{rank}\, [\tilde{Q}(t) - Q(t)] \text{ and } \operatorname{ind}\, [\tilde{Q}(t) - Q(t)] \text{ are constant on } [\alpha, \infty). \tag{1.18}$$

In particular, the inequality $\tilde{Q}(t) \geq Q(t)$ holds on $[\alpha, \infty)$ if and only if $\tilde{Q}(\alpha) \geq Q(\alpha)$, and the inequality $\tilde{Q}(t) > Q(t)$ holds on $[\alpha, \infty)$ if and only if $\tilde{Q}(\alpha) > Q(\alpha)$.

Theorem 1.13. *Assume (1.3) and let $\mathcal{G}$ be a genus of conjoined bases of (H) with the corresponding matrix $R_{\mathcal{G}}(t)$ in (1.9). Let $Q(t)$ be a symmetric solution of the Riccati equation ($\mathcal{R}$) on $[\alpha, \infty) \subseteq [\alpha_\infty, \infty)$ and let $\tilde{Q}(t)$ be a symmetric solution of ($\mathcal{R}$) satisfying the initial condition $\tilde{Q}(\alpha) \geq Q(\alpha)$. Then the matrix $\tilde{Q}(t)$ solves ($\mathcal{R}$) on the whole interval $[\alpha, \infty)$ such that the inequality $\tilde{Q}(t) \geq Q(t)$ holds for all $t \in [\alpha, \infty)$.*

Theorem 1.14. *Let $\mathcal{G}$ be a genus of conjoined bases of (H) and let $R_{\mathcal{G}}(t)$ be its corresponding matrix in (1.9). Moreover, let $Q(t)$ be a symmetric solution of ($\mathcal{R}$) on $[\alpha, \infty) \subseteq [\alpha_\infty, \infty)$ satisfying condition (1.16). Let $\beta \in [\alpha, \infty)$ and $K \in \mathbb{R}^{n \times n}$ be given and consider the solution $\tilde{Q}(t)$ of ($\mathcal{R}$) with $\tilde{Q}(\beta) = K$. Then the following statements are equivalent.*

- (i) *The matrix $\tilde{Q}(t)$ solves the Riccati equation ($\mathcal{R}$) on the whole interval $[\alpha, \infty)$ such that $R_{\mathcal{G}}(t)\, \tilde{Q}(t)\, R_{\mathcal{G}}(t) = Q(t)$ holds for every $t \in [\alpha, \infty)$.*
- (ii) *The matrix K satisfies the equality $R_{\mathcal{G}}(\beta)\, K\, R_{\mathcal{G}}(\beta) = Q(\beta)$.*

Let $\mathcal{G}$ be a genus of conjoined bases of (H) and let $R_{\mathcal{G}}(t)$ be its representing orthogonal projector in (1.9). For a given solution $Q(t)$ of the Riccati equation ($\mathcal{R}$) on a subinterval $[\alpha, \infty) \subseteq [\alpha_\infty, \infty)$ we consider the following system of first order linear differential equations

$$\left.\begin{aligned} \Theta' &= [\mathcal{A}(t) + \mathcal{B}(t)\, Q(t)]\, \Theta, \\ \Omega' &= \mathcal{A}(t)\, \Omega + [I - R_{\mathcal{G}}(t)]\{C(t) - [A(t) + A^T(t)]\, Q(t)\}\, \Theta, \end{aligned}\right\} \quad t \in [\alpha, \infty), \tag{1.19}$$

together with the initial conditions

$$\Theta(\alpha) = K, \quad \Omega(\alpha) = L, \tag{1.20}$$

where the matrices $K, L \in \mathbb{R}^{n \times n}$ satisfy

$$\operatorname{Im} K \subseteq \operatorname{Im} R_{\mathcal{G}}(\alpha), \quad \operatorname{Im} L \subseteq \operatorname{Ker} R_{\mathcal{G}}(\alpha), \quad \operatorname{rank}\, (K^T, L^T)^T = n. \tag{1.21}$$

The initial value problem (1.19)–(1.20) serves for the formulation of the main results of this section. The first equation in (1.19) is motivated by the approach in [68, Chapter 2, Lemma 2.1], which is adopted here to the setting of uncontrollable systems (H).

According to [77, Remark 4.8 and Proposition 4.9] initial value problem (1.19)–(1.20) with (1.21) has always the solution (Θ, Ω), which is unique up to a right nonsingular constant multiple. Moreover, the matrix $\Theta(t)$ has a constant kernel on $[\alpha, \infty)$ and

$$\operatorname{Im}\Theta(t) = \operatorname{Im} R_{\mathcal{G}}(t), \quad \operatorname{Im}\Omega(t) = \operatorname{Ker} R_{\mathcal{G}}(t), \quad \operatorname{rank}\,(\Theta^T(t), \Omega^T(t))^T = n, \quad t \in [\alpha, \infty). \tag{1.22}$$

These properties of the matrix $\Theta(t)$ allow us to define the function

$$F_\alpha(t) := \int_\alpha^t \Theta^\dagger(s)\,\mathcal{B}(s)\,\Theta^{\dagger T}(s)\,\mathrm{d}s, \quad t \in [\alpha, \infty), \tag{1.23}$$

which will be referred to as the *F-matrix* corresponding to the solution $Q(t)$ with respect to the genus $\mathcal{G}$. We note that for an invertible $\Theta(t)$ the matrix $F_\alpha(t)$ in (1.23) was considered in [68, Section 2.2]. Here we allow $\Theta(t)$ to be singular. By [77, Remark 4.11] it follows that $F_\alpha(t)$ is symmetric and the inclusion $\operatorname{Im} F_\alpha(t) \subseteq \operatorname{Im} R_{\mathcal{G}}(\alpha)$ holds for every $t \in [\alpha, \infty)$. Moreover, under (1.3) the matrix $F_\alpha(t)$ is nonnegative definite and nondecreasing and the limit

$$D_\alpha := \lim_{t\to\infty} F_\alpha^\dagger(t). \tag{1.24}$$

exists, where the matrix D_α is symmetric and nonnegative definite with $\operatorname{Im} D_\alpha \subseteq \operatorname{Im} R_{\mathcal{G}}(\alpha)$.

The next two results extend the well known correspondence between the symmetric solutions $Q(t)$ of the classical Riccati equation (R) on $[\alpha, \infty)$ and conjoined bases (X, U) of (H) with $X(t)$ invertible on $[\alpha, \infty)$, i.e.,

$$Q(t) = U(t)\,X^{-1}(t) \quad \text{on } [\alpha, \infty) \tag{1.25}$$

to the case of possibly noninvertible $X(t)$ on $[\alpha, \infty)$. For this purpose we utilize the Moore–Penrose generalized inverse $X^\dagger(t)$ of the matrix $X(t)$, called also the pseudoinverse, see e.g. [13], [14, Chapter 6], and [19, Section 1.4]. Given a conjoined basis (X, U) of (H) and a point $\alpha \in [a, \infty)$, the matrix $Q(t)$ defined by

$$Q(t) := X(t)\,X^\dagger(t)\,U(t)\,X^\dagger(t) = R(t)\,U(t)\,X^\dagger(t), \quad t \in [\alpha, \infty), \tag{1.26}$$

is called the *Riccati quotient* associated with the conjoined basis (X, U) on the interval $[\alpha, \infty)$. Here $R(t) := X(t)\,X^\dagger(t)$ is the orthogonal projector onto the subspace $\operatorname{Im} X(t)$ for all $t \in [\alpha, \infty)$. By [71, pg. 24] the matrix $Q(t)$ is symmetric and satisfies on $[\alpha, \infty)$ the properties

$$X^T(t)\,Q(t)\,X(t) = X^T(t)\,U(t), \quad \operatorname{Im} Q(t) \subseteq \operatorname{Im} R(t), \quad Q(t)\,X(t) = R(t)\,U(t). \tag{1.27}$$

In addition, if (X, U) has constant kernel on $[\alpha, \infty)$, then by [19, Theorems 10.5.1 and 10.5.3] the matrix $X^\dagger(t)$ is piecewise continuously differentiable on $[\alpha, \infty)$, and hence also the matrix $Q(t)$ is piecewise continuously differentiable on $[\alpha, \infty)$. Note that when $X(t)$ is an invertible matrix, then the Riccati quotient $Q(t)$ in (1.26) reduces to (1.25).

Theorem 1.15. *Let $\mathcal{G}$ be a genus of conjoined bases of (H) with the orthogonal projector $R_{\mathcal{G}}(t)$ in (1.9). Moreover, let (X, U) be a conjoined basis of (H) belonging to $\mathcal{G}$ such that (X, U) has constant kernel on a subinterval $[\alpha, \infty) \subseteq [\alpha_\infty, \infty)$ and let $Q(t)$ be the corresponding Riccati quotient in (1.26). Then the matrix $Q(t)$ is a symmetric solution of the Riccati equation ($\mathcal{R}$) on $[\alpha, \infty)$ such that the condition in (1.16) holds and the matrices $\Theta(t)$ and $\Omega(t)$ defined by*

$$\Theta(t) := X(t), \quad \Omega(t) := U(t) - Q(t)\,X(t), \quad t \in [\alpha, \infty), \tag{1.28}$$

solve the initial value problem (1.19)–(1.20) on $[\alpha, \infty)$ with (1.21).

Theorem 1.16. *Let $\mathcal{G}$ be a genus of conjoined bases of (H) with the orthogonal projector $R_{\mathcal{G}}(t)$ in (1.9) and let $Q(t)$ be a solution of the Riccati equation ($\mathcal{R}$) on $[\alpha, \infty) \subseteq [\alpha_\infty, \infty)$ such that the matrix $R_{\mathcal{G}}(t)\,Q(t)\,R_{\mathcal{G}}(t)$ is symmetric on $[\alpha, \infty)$. Moreover, let (Θ, Ω) be a solution of (1.19)–(1.20) on $[\alpha, \infty)$ with (1.21) and define the matrices*

$$X(t) := \Theta(t), \quad U(t) := Q(t)\,\Theta(t) + \Omega(t), \quad t \in [\alpha, \infty). \tag{1.29}$$

Then the following statements hold.

(i) *The pair (X, U) is a conjoined basis of (H) such that (X, U) has a constant kernel on $[\alpha, \infty)$ and belongs to the genus $\mathcal{G}$.*

(ii) *The matrix $R_{\mathcal{G}}(t)\, Q(t)\, R_{\mathcal{G}}(t)$ is the Riccati quotient in (1.26) associated with the conjoined basis (X, U) on $[\alpha, \infty)$, i.e., the equality $R_{\mathcal{G}}(t)\, Q(t)\, R_{\mathcal{G}}(t) = R(t)\, U(t)\, X^{\dagger}(t)$ holds for all $t \in [\alpha, \infty)$, where $R(t)$ is the corresponding orthogonal projector onto $\operatorname{Im} X(t)$.*

Let $\mathcal{G}$ be a genus of conjoined basis of (H) with the associated matrix $R_{\mathcal{G}}(t)$ in (1.9) and let $[\alpha, \infty) \subseteq [\alpha_\infty, \infty)$ be a given interval. The results in Theorems 1.15 and 1.16 provide a correspondence between the set of all conjoined basis (X, U) of (H) with constant kernel on $[\alpha, \infty)$, which belong to the genus $\mathcal{G}$, and the set of all symmetric solutions $Q(t)$ of the Riccati equation ($\mathcal{R}$) on $[\alpha, \infty)$ satisfying condition (1.16). More precisely, for every such a conjoined basis (X, U) its Riccati quotient $Q(t)$ in (1.26) is a symmetric solution of ($\mathcal{R}$) on $[\alpha, \infty)$ with (1.16). Conversely, if $Q(t)$ is a symmetric solution of ($\mathcal{R}$) on $[\alpha, \infty)$ satisfying (1.16), then there exists a conjoined basis (X, U) of (H) from the genus $\mathcal{G}$ with constant kernel on $[\alpha, \infty)$ such that $Q(t)$ is its corresponding Riccati quotient from (1.26).

The last part of this section is devoted to the implicit Riccati equations (1.14) and

$$R_{\mathcal{G}}(t)\, [Q' + Q\, \mathcal{A}(t) + \mathcal{A}^T(t)\, Q + Q\, \mathcal{B}(t)\, Q - \mathcal{C}(t)]\, R_{\mathcal{G}}(t) = 0, \quad [\alpha, \infty), \tag{1.30}$$

with $\alpha \in [\alpha_\infty, \infty)$. These implicit Riccati equations were used in [50, Section 6] in several criteria characterizing the nonnegativity and positivity of the associated quadratic functional.

Theorem 1.17. *Let $\mathcal{G}$ be a genus of conjoined bases of (H) and let $R_{\mathcal{G}}(t)$ be the corresponding orthogonal projector in (1.9). Moreover, let $Q(t)$ be a symmetric matrix defined on $[\alpha, \infty) \subseteq [\alpha_\infty, \infty)$ such that condition (1.16) holds. Then the following statements are equivalent.*

(i) *The matrix $Q(t)$ solves the Riccati equation ($\mathcal{R}$) on $[\alpha, \infty)$.*

(ii) *The matrix $Q(t)$ solves the implicit Riccati equation (1.30) on $[\alpha, \infty)$.*

(iii) *The matrix $Q(t)$ solves the implicit Riccati equation (1.14) on $[\alpha, \infty)$.*

The above result shows that under a certain assumption we can transfer the problem of solving the implicit Riccati matrix differential equations (1.30) and (1.14) into a problem of solving the explicit Riccati matrix differential equation ($\mathcal{R}$).

1.4. Distinguished solutions of Riccati equations

In this section we study, for a given genus $\mathcal{G}$, symmetric solutions of the Riccati equation ($\mathcal{R}$), which correspond to principal solutions of (H) at infinity belonging to the genus $\mathcal{G}$ (to be defined below). This correspondence is based on the results in Theorems 1.15 and 1.16. In particular, we establish the results about distinguished solutions of ($\mathcal{R}$) at infinity regarding their relationship to principal solutions at infinity and to the nonoscillation of system (H), their interval of existence, their mutual classification within the genus $\mathcal{G}$, and their minimality in a suitable sense.

It may be surprising that these results comply with the known theory of distinguished solutions of the Riccati equation (R) for a controllable system (H) only partially. In many aspects the presented theory for general uncontrollable system (H) is substantially different. This is related to the nature of the problem, since for each genus $\mathcal{G}$ of conjoined bases of (H) there is a different Riccati equation ($\mathcal{R}$), but even within one genus $\mathcal{G}$ there may be many distinguished solutions of ($\mathcal{R}$) at infinity. We discuss these issues in Remark 1.31 at the end of this section. We note that the true uniqueness and minimality of the distinguished solution of ($\mathcal{R}$) at infinity is satisfied only in the minimal genus $\mathcal{G}_{\min}$ (see Theorem 1.30).

The following definition extends the notion of a distinguished solution (also called a principal solution) of (R) at infinity for a controllable system (H) in [21, pg. 53].

Definition 1.18. Let $\mathcal{G}$ be a genus of conjoined bases of (H) with the orthogonal projector $R_{\mathcal{G}}(t)$ in (1.9). A symmetric solution $\hat{Q}(t)$ of the Riccati equation ($\mathcal{R}$) is said to be a *distinguished solution at infinity* if the matrix $\hat{Q}(t)$ is defined on an interval $[\alpha, \infty) \subseteq [\alpha_\infty, \infty)$ and its corresponding matrix $\hat{F}_\alpha(t)$ in (1.23) satisfies $\hat{F}_\alpha^{\dagger}(t) \to 0$ as $t \to \infty$.

The notion in Definition 1.18 also extends the distinguished solution of the Riccati equation (R) introduced by W. T. Reid in [66, Section IV] and [68, Section 2.7], which in our context corresponds to the maximal genus $\mathcal{G} = \mathcal{G}_{\max}$ (for which $R_{\mathcal{G}}(t) \equiv I$).

The main results of this section compare the properties of distinguished solutions at infinity of the Riccati equation ($\mathcal{R}$) with the properties of principal solutions of system (H) at infinity belonging to the genus $\mathcal{G}$. For this purpose we recall the definition of the latter object. Following [80, Definition 7.1], we say that a conjoined basis $(\hat{X}, \hat{U})$ of (H) is a *principal solution at infinity* if $(\hat{X}, \hat{U})$ has constant kernel on $[\alpha, \infty)$ and its corresponding matrix $\hat{S}_\alpha(t)$ defined by

$$\hat{S}_\alpha(t) := \int_\alpha^t \hat{X}^\dagger(s)\, B(s)\, \hat{X}^{\dagger T}(s)\, \mathrm{d}s, \quad t \in [\alpha, \infty), \tag{1.31}$$

satisfies $\hat{S}_\alpha^\dagger(t) \to 0$ as $t \to \infty$. In this case we will say that $(\hat{X}, \hat{U})$ is a principal solution of (H) at infinity with respect to the interval $[\alpha, \infty)$. By (1.13), the principal solutions of (H) can be classified according to the rank of $\hat{X}(t)$ on $[\alpha, \infty)$. In particular, the *minimal principal solution* $(\hat{X}_{\min}, \hat{U}_{\min})$ of (H) at infinity satisfies $\operatorname{rank} \hat{X}_{\min}(t) = n - d_\infty$, while the *maximal principal solution* $(\hat{X}_{\max}, \hat{U}_{\max})$ of (H) at infinity is determined by $\operatorname{rank} \hat{X}_{\max}(t) = n$, hence $\hat{X}_{\max}(t)$ is invertible on $[\alpha, \infty)$, see [80, Remark 7.2].

In the next proposition we recall from [80, Theorem 7.6] and [79, Theorems 7.6] the characterization of the nonoscillation of system (H) by the existence of a principal solution of (H) at infinity with any possible rank, as well as the uniqueness of the minimal principal solution.

Proposition 1.19. *Assume that (1.3) holds. Then the following statements are equivalent.*

(i) *System (H) is nonoscillatory.*
(ii) *There exists a principal solution of (H) at infinity.*
(iii) *For any integer r satisfying $n - d_\infty \le r \le n$ there exists a principal solution of (H) at infinity with rank equal to r.*

In particular, system (H) is nonoscillatory if and only if there exists a minimal principal solution of (H) at infinity. In this case the minimal principal solution is unique up to a right nonsingular constant multiple.

In [80, Equation 7.4] we defined for a nonoscillatory system (H) the point $\hat{\alpha}_{\min} \in [a, \infty)$ by

$$\hat{\alpha}_{\min} := \inf \big\{ \alpha \in [a, \infty),\ (\hat{X}_{\min}, \hat{U}_{\min}) \text{ has constant kernel on } [\alpha, \infty) \big\}, \tag{1.32}$$

where $(\hat{X}_{\min}, \hat{U}_{\min})$ is the minimal principal solution of (H) at infinity. We note that the equality $d[\alpha, \infty) = d_\infty$ holds for every $\alpha > \hat{\alpha}_{\min}$, see [80, Theorem 7.9]. In turn, combining this fact with formula (1.7) we obtain that

$$d[\hat{\alpha}_{\min}, \infty) = d_\infty, \quad \text{i.e.,} \quad \hat{\alpha}_{\min} \ge \alpha_\infty.$$

In the remaining part of this section we present the main results from [77, Sections 3 and 7], see Appendix B. The following two results show that in the context of Theorems 1.15 and 1.16 the distinguished solutions of the Riccati equation ($\mathcal{R}$) correspond to the principal solutions of (H) at infinity from the genus $\mathcal{G}$.

Theorem 1.20. *Let $\mathcal{G}$ be a genus of conjoined bases of (H) and $R_{\mathcal{G}}(t)$ be the orthogonal projector in (1.9). Moreover, let $\hat{Q}(t)$ be a distinguished solution of ($\mathcal{R}$) at infinity with respect to the interval $[\alpha, \infty) \subseteq [\alpha_\infty, \infty)$. Then every conjoined basis $(\hat{X}, \hat{U})$ of (H), which is associated with $\hat{Q}(t)$ on $[\alpha, \infty)$ via Theorem 1.16, is a principal solution of (H) at infinity with respect to $[\alpha, \infty)$ belonging to $\mathcal{G}$.*

Theorem 1.21. *Let $(\hat{X}, \hat{U})$ be a principal solution of (H) at infinity with respect to the interval $[\alpha, \infty)$, which belongs to a genus $\mathcal{G}$. Moreover, let $\hat{Q}(t)$ be the Riccati quotient in (1.26) associated with $(\hat{X}, \hat{U})$ on $[\alpha, \infty)$. Then $\hat{Q}(t)$ is a distinguished solution of the Riccati equation ($\mathcal{R}$) at infinity with respect to $[\alpha, \infty)$.*

From Theorems 1.20 and 1.21 it follows that the property of the existence of a principal solution of (H) at infinity in the genus $\mathcal{G}$, as stated in [80, Theorem 7.12], transfers naturally to the existence of a distinguished solution of the associated Riccati equation ($\mathcal{R}$).

Corollary 1.22. *Let $\mathcal{G}$ be a genus of conjoined bases of (H) with the orthogonal projector $R_{\mathcal{G}}(t)$ in (1.9). Then there exists a principal solution of (H) at infinity belonging to the genus $\mathcal{G}$ if and only if there exists a distinguished solution of the Riccati equation ($\mathcal{R}$) at infinity. In this case, the set of all Riccati quotients in (1.26), which correspond to the principal solutions $(\hat{X}, \hat{U})$ of (H) at infinity from the genus $\mathcal{G}$, coincides with the set of all matrices $R_{\mathcal{G}}\, \hat{Q}\, R_{\mathcal{G}}$, where $\hat{Q}$ is a distinguished solution of ($\mathcal{R}$) at infinity.*

In the following result we characterize the nonoscillation of system (H) in terms of the existence of a distinguished solution of ($\mathcal{R}$) in a given (or every) genus $\mathcal{G}$. This corresponds to Proposition 1.19 regarding the principal solutions of (H) at infinity.

Theorem 1.23. *Assume that (1.3) holds. Then the following statements are equivalent.*

(i) *System (H) is nonoscillatory.*
(ii) *There exists a distinguished solution of the Riccati equation ($\mathcal{R}$) for some genus $\mathcal{G}$.*
(iii) *There exists a distinguished solution of the Riccati equation ($\mathcal{R}$) for every genus $\mathcal{G}$.*

The result in Theorem 1.23 justifies the development of the theory of genera of conjoined bases for possibly oscillatory system (H) in Section 1.2. Of course, assuming that system (H) is nonoscillatory, then it is sufficient to use the theory of genera of conjoined bases from [80, Section 6] and [82, Section 4] for the construction of distinguished solutions of the Riccati equation ($\mathcal{R}$) for a genus $\mathcal{G}$. It is the converse to this implication, which requires a more general approach, since in this case we need to define the coefficients of equation ($\mathcal{R}$) without the assumption of nonoscillation of system (H).

In the following result we present a mutual classification of all distinguished solutions of ($\mathcal{R}$). This classification is formulated in terms of the initial values of the involved distinguished solutions at some point α from the maximal interval $(\hat{\alpha}_{\min}, \infty)$.

Theorem 1.24. *Assume that (1.3) holds and system (H) is nonoscillatory with $\hat{\alpha}_{\min}$ and $R_{\Lambda\infty}(t)$ defined in (1.32) and (1.8), respectively. Let $\mathcal{G}$ be a genus of conjoined bases of (H) and let $R_{\mathcal{G}}(t)$ be the matrix in (1.9). Moreover, let $\hat{Q}(t)$ be a distinguished solution of the Riccati equation ($\mathcal{R}$) at infinity. Then a symmetric solution $Q(t)$ of ($\mathcal{R}$) defined on a neighborhood of some point $\alpha \in (\hat{\alpha}_{\min}, \infty)$ is a distinguished solution at infinity if and only if*

$$R_{\Lambda\infty}(\alpha)\, Q(\alpha)\, R_{\Lambda\infty}(\alpha) = R_{\Lambda\infty}(\alpha)\, \hat{Q}(\alpha)\, R_{\Lambda\infty}(\alpha). \tag{1.33}$$

In the next three results we study the minimality of distinguished solutions of ($\mathcal{R}$). This minimality property needs to be understood in the following sense. For every symmetric solution $Q(t)$ of ($\mathcal{R}$) there exists a distinguished solution of ($\mathcal{R}$), which exists on the same interval and is at the same time smaller than $Q(t)$ on this interval (Theorems 1.25 and 1.26). On the other hand, any symmetric solution of (H), which is smaller than a distinguished solution of (H) on some interval, is a distinguished solution itself with respect to this interval (Theorem 1.27). However, in general there is no universal "smallest" distinguished solution of ($\mathcal{R}$), see Remark 1.28 below. We note that in the first result we consider the case when the solutions satisfy condition (1.16), while in the second and third result this assumption is removed.

Theorem 1.25. *Assume (1.3). Let $\mathcal{G}$ be a genus of conjoined bases of (H) with the matrix $R_{\mathcal{G}}(t)$ in (1.9) and let $Q(t)$ be a symmetric solution of the Riccati equation ($\mathcal{R}$) on $[\alpha, \infty) \subseteq [\alpha_\infty, \infty)$ such that inclusion (1.16) holds. Then there exists a distinguished solution $\hat{Q}(t)$ of ($\mathcal{R}$) at infinity with respect to $[\alpha, \infty)$ satisfying(1.16) such that $Q(t) \geq \hat{Q}(t)$ for every $t \in [\alpha, \infty)$.*

Theorem 1.26. *Assume (1.3). Let $\mathcal{G}$ be a genus of conjoined bases of (H) with the orthogonal projector $R_{\mathcal{G}}(t)$ in (1.9). Let $Q(t)$ be a symmetric solution of the Riccati equation ($\mathcal{R}$) on $[\alpha, \infty) \subseteq [\alpha_\infty, \infty)$. Then there exists a distinguished solution $\hat{Q}(t)$ of ($\mathcal{R}$) at infinity with respect to $[\alpha, \infty)$ such that the inequality $Q(t) \geq \hat{Q}(t)$ holds for every $t \in [\alpha, \infty)$.*

We note that the converse to Theorem 1.26 also holds. More precisely, if $\hat{Q}(t)$ is a distinguished solution of ($\mathcal{R}$) at infinity with respect to the interval $[\alpha, \infty)$, then every symmetric solution $Q(t)$ of ($\mathcal{R}$), which satisfies the condition $Q(\alpha) \geq \hat{Q}(\alpha)$, exists on the whole interval $[\alpha, \infty)$ and the inequality $Q(t) \geq \hat{Q}(t)$ holds for every $t \in [\alpha, \infty)$. This observation is a direct application of Theorem 1.13 with $Q := \hat{Q}$ and $\tilde{Q} := Q$.

Theorem 1.27. *Assume* (1.3) *and let $\mathcal{G}$ be a genus of conjoined bases of* (H) *with the matrix $R_{\mathcal{G}}(t)$ in* (1.9)*. Let $\tilde{Q}(t)$ be a distinguished solution of the Riccati equation* ($\mathcal{R}$) *with respect to the interval $[\alpha, \infty) \subseteq [\alpha_\infty, \infty)$. Moreover, let $Q(t)$ be a symmetric solution of* ($\mathcal{R}$) *on $[\alpha, \infty)$ satisfying the initial condition $\tilde{Q}(\alpha) \geq Q(\alpha)$. Then $Q(t)$ is a distinguished solution of* ($\mathcal{R}$) *at infinity with respect to $[\alpha, \infty)$ and the inequality $\tilde{Q}(t) \geq Q(t)$ holds for all $t \in [\alpha, \infty)$.*

Remark 1.28. Given a genus $\mathcal{G}$ of conjoined bases of (H) with the matrix $R_{\mathcal{G}}(t)$ defined in (1.9), let $\hat{Q}(t)$ be a distinguished solution of ($\mathcal{R}$) at infinity with respect to the interval $[\alpha, \infty) \subseteq [\alpha_\infty, \infty)$. Then there exist distinguished solutions $\hat{Q}_*(t)$ and $\hat{Q}_{**}(t)$ of ($\mathcal{R}$) satisfying

$$\hat{Q}_*(t) \leq \hat{Q}(t) \leq \hat{Q}_{**}(t), \quad t \in [\alpha, \infty). \tag{1.34}$$

The solutions $\hat{Q}_*(t)$ and $\hat{Q}_{**}(t)$ are given, for example, by the initial conditions

$$\hat{Q}_*(\alpha) = \hat{Q}(\alpha) - I + R_{\Lambda\infty}(\alpha) \quad \text{and} \quad \hat{Q}_{**}(\alpha) = \hat{Q}(\alpha) + I - R_{\Lambda\infty}(\alpha), \tag{1.35}$$

where $R_{\Lambda\infty}(t)$ is the orthogonal projector defined in (1.8). Therefore, for the case of a general (not necessarily controllable) system (H) the partially ordered set of all distinguished solutions of ($\mathcal{R}$) has neither a minimal element nor a maximal element.

The considerations in Theorems 1.24 and 1.25 show that for the minimal genus $\mathcal{G}_{\min}$, i.e., for $R_{\mathcal{G}}(t) = R_{\Lambda\infty}(t)$, there exists a uniquely determined distinguished solution of ($\mathcal{R}$) with

$$\left.\begin{aligned} \mathcal{A}(t) &:= A(t) R_{\Lambda\infty}(t) - A^T(t)\,[I - R_{\Lambda\infty}(t)], \\ \mathcal{B}(t) &:= B(t), \\ \mathcal{C}(t) &:= R_{\Lambda\infty}(t)\, C(t)\, R_{\Lambda\infty}(t), \end{aligned}\right\} \quad t \in [\alpha_\infty, \infty), \tag{1.36}$$

which is the smallest element in the set of all symmetric solutions $Q(t)$ of ($\mathcal{R}$) satisfying (1.16).

Definition 1.29. Let $\mathcal{G}_{\min}$ be the minimal genus of conjoined bases of (H) with the minimal orthogonal projector $R_{\Lambda\infty}(t)$ in (1.8). A symmetric solution $\hat{Q}(t)$ of the Riccati equation ($\mathcal{R}$) with the coefficients in (1.36) is said to be a *minimal distinguished solution at infinity* if the matrix $\hat{Q}(t)$ is defined on an interval $[\alpha, \infty) \subseteq [\alpha_\infty, \infty)$ such that

$$\operatorname{Im} \hat{Q}(t) \subseteq \operatorname{Im} R_{\Lambda\infty}(t), \quad t \in [\alpha, \infty), \tag{1.37}$$

and its corresponding matrix $\hat{F}_\alpha(t)$ in (1.23) satisfies $\hat{F}_\alpha^\dagger(t) \to 0$ as $t \to \infty$.

The following result shows the existence and uniqueness of the minimal distinguished solution of ($\mathcal{R}$) for the minimal genus $\mathcal{G}_{\min}$, as well as its minimality property.

Theorem 1.30. *Assume* (1.3)*. Then system* (H) *is nonoscillatory if and only if there exists a minimal distinguished solution $\hat{Q}(t)$ of the Riccati equation* ($\mathcal{R}$) *with the coefficients in* (1.36)*. In this case, the minimal distinguished solution $\hat{Q}(t)$ is determined uniquely and any symmetric solution $Q(t)$ of* ($\mathcal{R}$) *on $[\alpha, \infty) \subseteq [\alpha_\infty, \infty)$ with* (1.37) *satisfies $Q(t) \geq \hat{Q}(t)$ on $[\alpha, \infty)$.*

The minimal distinguished solution of ($\mathcal{R}$) at infinity in Theorem 1.30 will be denoted by $\hat{Q}_{\min}$. The minimal distinguished solution $\hat{Q}_{\min}$ plays for the theory of Riccati differential equations ($\mathcal{R}$) or (R) a similar role as the minimal principal solution $(\hat{X}_{\min}, \hat{U}_{\min})$ of (H) at infinity for the theory of principal solutions at infinity.

Remark 1.31. When system (H) is completely controllable, the main results of this section give the classical statements about the distinguished solutions at infinity of the Riccati equation (R). More precisely, the following holds.

- The results in Corollary 1.22 and Theorem 1.24 yield the correspondence between the unique principal solution of (H) at infinity and the unique distinguished solution of (R) at infinity, see [21, pg. 53] or [68, pp. 45–46].
- The result in Theorem 1.23 provides a characterization of the nonoscillation of system (H) in terms of the existence of the unique distinguished solution of (R) at infinity, see the necessary condition in [67, Theorem VII.3.3]. Note that the nonoscillation of (H) is defined in [67, Section VII.3] in terms of disconjugacy of (H), i.e., in terms of the nonexistence of mutually conjugate points, which is a stronger concept than the nonoscillation of (H). We note also that the sufficiency part of Theorem 1.23 is new also in the completely controllable case.
- The results in Theorems 1.26 and 1.27 yield the minimality property of the unique distinguished solution of (R) at infinity, see [21, Theorem 8, pg. 54] or [68, Theorem IV.4.2].

Indeed, in this case $d_\infty = 0$ and there is only one minimal/maximal genus of conjoined bases of (H). This implies that $\alpha_\infty = a$ and the orthogonal projector $R_{\Lambda\infty}(t)$ in (1.8) satisfies $R_{\Lambda\infty}(t) \equiv I$ on $[a,\infty)$. Therefore, the unique Riccati equation ($\mathcal{R}$) associated with the minimal/maximal genus coincides with the classical Riccati equation (R). Moreover, under the Legendre condition (1.3) the nonoscillation of system (H) is then equivalent with the existence of a unique (minimal) distinguished solution $\hat{Q}$ of (R) at infinity. In addition, the matrix $\hat{Q}$ constitutes the smallest symmetric solution of the Riccati equation (R), that is, every symmetric solution Q of (R) on $[\alpha,\infty) \subseteq [a,\infty)$ satisfies inequality (1.17).

1.5. Riccati quotients and comparative index

The Riccati quotient $Q(t)$ defined in (1.26), resp. in (1.25), represents an important tool in the investigations related to the Sturmian theory of system (H). In particular, the results in Section 2.1 show that the changes in the index (i.e., the number of negative eigenvalues) of the difference of two such Riccati quotients determine the difference between the numbers of focal points of two conjoined bases of a completely controllable system (H).

A mathematical tool, which describes in full generality such behavior, is the comparative index invented by Elyseeva [32,33]. The *comparative index* $\mu(Y,\tilde{Y})$ and the *dual comparative index* $\mu^*(Y,\tilde{Y})$ of two constant real $2n\times n$ matrices $Y=(X,U)$ and $\tilde{Y}=(\tilde{X},\tilde{U})$ satisfying

$$Y^T\mathcal{J}\,Y=0, \quad \tilde{Y}^T\mathcal{J}\,\tilde{Y}=0, \quad \operatorname{rank} Y = n = \operatorname{rank}\tilde{Y} \tag{1.38}$$

are nonnegative integers (between 0 and n) defined by the equations

$$\mu(Y,\tilde{Y}) := \operatorname{rank}\mathcal{M} + \operatorname{ind}\mathcal{P}, \quad \mu^*(Y,\tilde{Y}) := \operatorname{rank}\mathcal{M} + \operatorname{ind}(-\mathcal{P}), \tag{1.39}$$

where

$$\mathcal{M} := (I - X^\dagger X)\,W(Y,\tilde{Y}), \quad \mathcal{P} := V\,[W(Y,\tilde{Y})]^T X^\dagger \tilde{X} V, \quad V := I - \mathcal{M}^\dagger\mathcal{M} \tag{1.40}$$

are $n\times n$ matrices with $W(Y,\tilde{Y}) := Y^T\mathcal{J}\,\tilde{Y}$ being the Wronskian of Y and $\tilde{Y}$. The $2n\times 2n$ matrix $\mathcal{J}$ is defined in (1.1). Here $\operatorname{ind}\mathcal{P}$ denotes the index (i.e., the number of negative eigenvalues) of the symmetric matrix $\mathcal{P}$. Originally, the comparative index was developed for the discrete oscillation theory (see below) due to its connection with the "discrete focal points" [27, 32, 33]. This new approach allowed to solve several difficult open problems pertaining e.g. exact Sturmian separation and comparison theorems on compact interval (Section 2.2), a detailed distribution of focal points of conjoined bases throughout the given interval (Theorem 2.8), singular Sturmian separation and comparison theorems on unbounded intervals (Section 2.3), or the development of the oscillation theory of system (H) without the Legendre condition (1.3) in [38–40]. Comparative index also produced several important generalizations in the spectral theory [37–39].

The definition of the comparative index shows that it can be expressed in terms of the Riccati quotients as in (1.27), which are associated with the matrices Y and $\tilde{Y}$. More precisely, let Q and $\tilde{Q}$ be any symmetric $n\times n$ matrices such that

$$X^TQX = X^TU, \quad \tilde{X}^T\tilde{Q}\tilde{X} = \tilde{X}^T\tilde{U}. \tag{1.41}$$

For example, according to (1.26) we can choose the symmetric matrices

$$Q := XX^{\dagger}UX^{\dagger}, \quad \tilde{Q} := \tilde{X}\tilde{X}^{\dagger}\tilde{U}\tilde{X}^{\dagger}. \tag{1.42}$$

Then the matrix $\mathcal{P}$ in (1.40) has the form

$$\mathcal{P} = V\tilde{X}^{T}(\tilde{Q} - Q)\tilde{X}V, \tag{1.43}$$

see e.g. [27, Theorem 3.2(iii)]. In particular, if the matrices X and $\tilde{X}$ are invertible, then $\mathcal{M} = 0$, $V = I$, and equations (1.39), (1.42), and (1.43) yield that

$$\mu(Y, \tilde{Y}) = \operatorname{ind}(\tilde{Q} - Q), \quad \mu^{*}(Y, \tilde{Y}) = \operatorname{ind}(Q - \tilde{Q}), \quad Q := UX^{-1}, \quad \tilde{Q} := \tilde{U}\tilde{X}^{-1}. \tag{1.44}$$

In the next chapter we will demonstrate the utility of the comparative index and the dual comparative index for the development of a precise Sturmian theory of system (H).

CHAPTER 2

Sturmian theory for linear Hamiltonian systems

Oscillation theory of linear Hamiltonian systems and Sturm–Liouville differential equations represents a classical topic in the qualitative theory of differential equations. Standard references include the monographs [**11**, **21**, **30**, **46**, **58**, **67**, **69**] by Atkinson, Coppel, Elias, Hartman, Kratz, and Reid, or more recently [**9**, **56**, **70**] by Amrein et al., Johnson, Obaya, Novo, Nũnez, Fabbri, and Rofe-Beketov and Kholkin. In this chapter we present an overview of the Sturmian separation and comparison theorems for linear Hamiltonian systems and our recent contributions to this subject.

Linear Hamiltonian systems (H) without the complete controllability assumption are intensively studied in the literature. For example, Johnson, Novo, Nũnez, and Obaya proved in [**54**, Theorem 3.6] a formula connecting the rotation number of system (H) with the number of left proper focal points of a conjoined basis (X, U) of (H) in $(a, b]$ when $b \to \infty$. Uncontrollable systems (H) were also considered in [**41**, **53**, **55**] in the relation with the notion of a weak disconjugacy of (H) and dissipative control processes, and in [**66**, **76**, **79**–**83**] when studying the principal solutions of (H) at infinity. Let us mention that the transformation theory of linear Hamiltonian systems developed by Došlý and Elyseeva in [**23**–**25**, **35**, **36**] is an important tool in the investigations of their qualitative properties, both in the context of the controllable and uncontrollable systems.

In this chapter we consider nonoscillatory linear Hamiltonian systems, as defined in [**92**]. If the systems are oscillatory, then it is possible to measure a comparison of two such systems by means of the concept of relative oscillation. For completely controllable systems it was developed by Došlý in [**26**], which was extended to possibly uncontrollable systems by Elyseeva in [**38**].

2.1. Review of Sturmian theory for controllable systems

Regarding the compact interval $\mathcal{I} = [a, b]$, basic Sturmian separation and comparison theorems for the second order Sturm–Liouville differential equations are presented in [**46**, Theorem XI.3.1] or [**69**, Theorem II.3.2(a)]. An extension of these results to completely controllable linear Hamiltonian systems (H) was derived in [**20**, Theorem 4] by Coppel, in [**10**, pg. 252] by Arnold (also quoted in [**70**, Theorem 4.8]), and in [**58**, Section 7.3] by Kratz. The notion of a completely controllable system (H) was defined in Section 1.1. In particular, we remark that Kratz proved in [**58**, Theorem 4.1.3] the following result.

Proposition 2.1. *Assume that the Legendre condition (1.3) holds on the interval $\mathcal{I}$. Then system (H) is completely controllable on $\mathcal{I}$ if and only if for every conjoined basis (X, U) of (H) the matrix $X(t)$ is singular only at isolated points in the interval $\mathcal{I}$.*

As it is demonstrated in the above proposition, the development of the classical Sturmian theory for linear Hamiltonian system (H) is based on two assumptions. Namely,

- the validity of the Legendre condition (1.3) on the interval $[a, b]$ and
- the complete controllability (or the identical normality) of system (H) on $[a, b]$.

The above result justifies the following definition. A point $t_0 \in \mathcal{I}$ is a *focal point* of a conjoined basis (X, U) of (H) if $X(t_0)$ is singular, and then

$$m(t_0) := \operatorname{def} X(t_0) = \dim \operatorname{Ker} X(t_0), \quad m(t_0) \leq n, \tag{2.1}$$

is its multiplicity. Here we use the terminology defect of a matrix (denoted by def) for the dimension of its kernel. For convenience we denote by $m(\mathcal{I})$ and $\widehat{m}(\mathcal{I})$ the total number of focal points of the conjoined bases (X, U) and $(\hat{X}, \hat{U})$ of (H) in the interval $\mathcal{I}$, that is,

$$m(\mathcal{I}) := \sum_{t\in\mathcal{I}} m(t), \quad \widehat{m}(\mathcal{I}) := \sum_{t\in\mathcal{I}} \widehat{m}(t), \tag{2.2}$$

where $\widehat{m}(t_0)$ is the multiplicity of the focal point t_0 of $(\hat{X}, \hat{U})$ defined according to (2.1). Under the Legendre condition (1.3) the sums in (2.2) are finite when the interval $\mathcal{I}$ is compact or when the interval $\mathcal{I}$ is unbounded from above and system (H) is nonoscillatory. If we deal with an interval $\mathcal{I}$ with specific endpoints, such as the intervals $\mathcal{I} = (a, b]$ or $[a, b)$ or (a, b) or (a, ∞) etc., then we write $m(a, b]$ or $\widehat{m}(a, b]$ etc. for simplicity in the corresponding context.

2.1.1. Sturmian separation theorems. In [67, Corollary 1, pg. 366] or [69, Corollary 1, pg. 306], Reid proved the following Sturmian separation theorem for a completely controllable system (H) on an arbitrary bounded interval $\mathcal{I}$. The numbers of focal points in the interval $\mathcal{I}$ of any two conjoined bases (X, U) and $(\hat{X}, \hat{U})$ of system (H) differ by at most n (which is the maximal multiplicity of a focal point), i.e.,

$$\big| m(\mathcal{I}) - \widehat{m}(\mathcal{I}) \big| \le n. \tag{2.3}$$

Moreover, an improved estimate was derived in [69, Corollary 3, pp. 307–308] saying that

$$\big| m(\mathcal{I}) - \widehat{m}(\mathcal{I}) \big| \le n - m, \tag{2.4}$$

where m is the defect (i.e., the dimension of the kernel) of the Wronskian of the two conjoined bases (X, U) and $(\hat{X}, \hat{U})$. In addition, for one conjoined basis (X, U) of (H) the difference between the numbers of its focal points in $(a, b]$ and in $[a, b)$ equals to the value

$$\operatorname{def} X(b) - \operatorname{def} X(a) = \operatorname{rank} X(a) - \operatorname{rank} X(b). \tag{2.5}$$

Specific results were also obtained for the principal solution (X_s, U_s) of system (H) at the point $s \in \mathcal{I}$, which we defined in (1.12). In this case we denote the number of focal points of (X_s, U_s) in the interval $\mathcal{I}$ by $m_s(\mathcal{I})$ in the spirit of (2.2). Then the result in [67, Corollary 2, pg. 366] or [69, Corollary 2, pg. 307] states that for any conjoined basis (X, U) of system (H) we have

$$m_a(a, b) - n \le m(a, b) \le m_a(a, b) + n, \tag{2.6}$$

$$m_a(a, b] - n \le m(a, b] \le m_a(a, b] + n, \tag{2.7}$$

while from [69, Theorem 8.3] we obtain that

$$m_a(a, b) = m_b(a, b), \quad m_a(a, b] = m_b[a, b). \tag{2.8}$$

As a continuation of the above results, Kratz derived in [58, Section 7.3] exact formulas for the difference of focal points in the open interval $\mathcal{I} = (a, b)$ of two conjoined bases (X, U) and $(\hat{X}, \hat{U})$ of system (H) by using the index (i.e., the number of negative eigenvalues) of the difference between the associated Riccati quotients. Namely, the results in [58, Theorem 7.3.1, pg. 194] states that

$$m(a, b) - \widehat{m}(a, b) = \operatorname{ind}(\hat{Q} - Q)(b^-) - \operatorname{ind}(\hat{Q} - Q)(a^+), \tag{2.9}$$

$$m(a, b) - \widehat{m}(a, b) = \operatorname{ind}(Q - \hat{Q})(a^+) - \operatorname{ind}(Q - \hat{Q})(b^-), \tag{2.10}$$

where $Q(t) := U(t)\, X^{-1}(t)$ and $\hat{Q}(t) := \hat{U}(t)\, \hat{X}^{-1}(t)$ according to (1.25) and where $\operatorname{ind}(\hat{Q} - Q)(t_0^{\pm})$ denote the one-sided limits of the index of the matrix $\hat{Q}(t) - Q(t)$. Note that $\operatorname{ind}[\hat{Q}(t) - Q(t)]$ is a piecewise constant quantity in the interval (a, b) under the Legendre condition (1.3) and that the difference on the right-hand side of (2.9) and (2.10) is always less or equal to n, which complies with the earlier estimate by Reid in (2.3). Moreover, by using the principal solutions at a and b, Kratz obtained in [58, Corollary 7.3.2, pg. 196] that for any conjoined basis (X, U) of (H) we have

$$m(a, b) = m_a(a, b) + \operatorname{ind}(Q_a - Q)(b^-) \ge m_a(a, b), \tag{2.11}$$

$$m(a, b) = m_b(a, b) + \operatorname{ind}(Q - Q_b)(a^+) \ge m_b(a, b), \tag{2.12}$$

where $Q_s(t) := U_s(t)\, X_s^{-1}(t)$.

A singular Sturmian separation theorem for a completely controllable and nonoscillatory system (H) on $\mathcal{I} = [a, \infty)$ was derived in [29] by Došlý and Kratz. The main ingredient is the concept of the principal solution of system (H) at infinity, which we discussed in Section 1.4. Following (1.31), the principal solution of (H) at infinity is defined as a conjoined basis (X_∞, U_∞) such that $X_\infty(t)$ is invertible on an interval $[\alpha, \infty)$ for some $\alpha \geq a$ and the matrix

$$T_\infty = 0, \quad \text{where } \; T_\infty := \lim_{t\to\infty} \left(\int_\alpha^t X_\infty^{-1}(s)\, B(s)\, X_\infty^{T-1}(s)\, \mathrm{d}s \right)^{-1}. \tag{2.13}$$

The principal solution (X_∞, U_∞) at infinity exists and is unique (up to a constant right invertible multiple) by Proposition 1.19. Then in [29, Theorem 1] it is shown that, under the Legendre condition (1.3), for any conjoined basis (X, U) of (H) we have the estimate

$$m[a, \infty) \geq m_\infty[a, \infty), \tag{2.14}$$

where $m_\infty[a, \infty)$ denotes the number of focal points of the (unique) principal solution of (H) at infinity in the indicated interval $[a, \infty)$. Moreover, the result in [29, Corollary 1] states that if the interval $\mathcal{I} = \mathbb{R} = (-\infty, \infty)$, then the numbers of focal points of the principal solutions of (H) at infinity and at minus infinity in the whole interval $\mathbb{R}$ satisfy

$$m_\infty(-\infty, \infty) = m_{-\infty}(-\infty, \infty), \tag{2.15}$$

which is a singular version of the first equality in (2.8). Moreover, by taking the limit for $a \to -\infty$ in (2.14) we obtain for any conjoined basis (X, U) of (H) the estimate

$$m(-\infty, \infty) \geq m_\infty(-\infty, \infty). \tag{2.16}$$

2.1.2. Sturmian comparison theorems. Next we review the Sturmian comparison theorems, which provide estimates for the numbers of focal points of two conjoined bases of two possibly different completely controllable linear Hamiltonian systems. Thus, together with system (H) we consider another linear Hamiltonian system

$$\hat{y}' = \mathcal{J}\hat{\mathcal{H}}(t)\, \hat{y}, \quad t \in \mathcal{I}, \tag{$\hat{\mathrm{H}}$}$$

where the coefficient matrix $\hat{\mathcal{H}} : \mathcal{I} \to \mathbb{R}^{2n\times 2n}$ is piecewise continuous and symmetric on $\mathcal{I}$. Moreover, we assume that the matrices $\mathcal{H}(t)$ in system (H) and $\hat{\mathcal{H}}(t)$ in system ($\hat{\mathrm{H}}$) are related by the Sturmian majorant condition

$$\mathcal{H}(t) \geq \hat{\mathcal{H}}(t) \quad \text{for all } t \in \mathcal{I}. \tag{2.17}$$

In this setting we say that system (H) is a Sturmian majorant of ($\hat{\mathrm{H}}$), or that system ($\hat{\mathrm{H}}$) is a Sturmian minorant of (H). In addition to (2.17) we assume that the minorant system ($\hat{\mathrm{H}}$) satisfies the Legendre condition

$$\hat{B}(t) \geq 0 \quad \text{for all } t \in \mathcal{I}. \tag{2.18}$$

Here $\hat{B}(t)$ is the lower right $n \times n$ block of $\hat{\mathcal{H}}(t)$, i.e., we partition the matrix $\hat{\mathcal{H}}(t)$ similarly to (1.1) as

$$\hat{\mathcal{H}}(t) = \begin{pmatrix} -\hat{C}(t) & \hat{A}^T(t) \\ \hat{A}(t) & \hat{B}(t) \end{pmatrix}, \quad t \in \mathcal{I}, \tag{2.19}$$

where $\hat{A}(t)$, $\hat{B}(t)$, $\hat{C}(t)$ are piecewise continuous $n \times n$ matrix-valued functions on $\mathcal{I}$ with $\hat{B}(t)$ and $\hat{C}(t)$ being symmetric for $t \in \mathcal{I}$. Assumption (2.17) then implies that the Legendre condition (1.3) holds also for the majorant system (H). Given a conjoined basis $(\hat{X}, \hat{U})$ of system ($\hat{\mathrm{H}}$), we denote by $\widehat{m}(\mathcal{I})$ the total number of its focal points in the interval $\mathcal{I}$, following the notation in (2.2).

In [20, Theorem 4], Coppel derived under assumptions (2.17) and (2.18) a comparison result on a bounded interval $\mathcal{I}$ for the principal solutions (X_a, U_a) and $(\hat{X}_a, \hat{U}_a)$ of systems (H) and ($\hat{\mathrm{H}}$) at the point a in the form

$$m_a(a, b] \geq \widehat{m}_a(a, b]. \tag{2.20}$$

Such estimates are also known in the works by Arnold in [10, pg. 252], which is also quoted in [70, Theorem 4.8] by Roffe-Beketov and Kholkin. However, the latter two references use the majorant condition (2.17) together with the strengthened Legendre condition $\hat{B}(t) > 0$ on the interval $\mathcal{I}$. On the other hand, in [58, Theorem 7.3.1, pg. 194], Kratz derived the Sturmian comparison theorem for the open interval $\mathcal{I} = (a, b)$ saying that for any two conjoined bases (X, U) and $(\hat{X}, \hat{U})$ of systems (H) and ($\hat{\text{H}}$) we have the estimates

$$m(a,b) - \widehat{m}(a,b) \geq \operatorname{ind}(\hat{Q} - Q)(b^-) - \operatorname{ind}(\hat{Q} - Q)(a^+), \tag{2.21}$$

$$m(a,b) - \widehat{m}(a,b) \geq \operatorname{ind}(Q - \hat{Q})(a^+) - \operatorname{ind}(Q - \hat{Q})(b^-), \tag{2.22}$$

where $Q(t) := U(t)\, X^{-1}(t)$ and $\hat{Q}(t) := \hat{U}(t)\, \hat{X}^{-1}(t)$ as we discussed above. Moreover, by using the principal solutions at a and b of system ($\hat{\text{H}}$) in the place of $(\hat{X}, \hat{U})$, Kratz obtained in [58, Corollary 7.3.2, pg. 196] that for any conjoined basis (X, U) of (H) we have the estimates

$$m(a,b) \geq \widehat{m}_a(a,b) + \operatorname{ind}(\hat{Q}_a - Q)(b^-) \geq \widehat{m}_a(a,b), \tag{2.23}$$

$$m(a,b) \geq \widehat{m}_b(a,b) + \operatorname{ind}(Q - \hat{Q}_b)(a^+) \geq \widehat{m}_b(a,b), \tag{2.24}$$

where $\hat{Q}_s(t) := \hat{U}_s(t)\, \hat{X}_s^{-1}(t)$. Note that according to the Sturmian separation theorem in (2.8) applied to system ($\hat{\text{H}}$) we have $\widehat{m}_b(a,b) = \widehat{m}_a(a,b)$ in (2.24). By taking $(X, U) := (X_a, U_a)$ being the principal solution of system (H) at a we obtain from (2.23) and (2.24) the inequality

$$m_a(a,b) \geq \widehat{m}_a(a,b), \tag{2.25}$$

which complements the earlier estimate in (2.20) by Coppel.

Regarding an open or unbounded interval $\mathcal{I}$, a singular Sturmian comparison theorem for the second order Sturm–Liouville differential equations was obtained in [1, Theorem 1(i)] by Aharonov and Elias. Moreover, a singular comparison theorem for completely controllable and nonoscillatory systems (H) and ($\hat{\text{H}}$) on $\mathcal{I} = [a, \infty)$ was derived in [29, Theorem 2] by Došlý and Kratz. More precisely, for any conjoined basis (X, U) of (H) we have the estimate

$$m[a,\infty) \geq \widehat{m}_\infty[a,\infty), \tag{2.26}$$

where $\widehat{m}_\infty[a,\infty)$ denotes the number of focal points of the (unique) principal solution of the minorant system ($\hat{\text{H}}$) at infinity. Moreover, if assumptions (2.17) and (2.18) hold on the unbounded interval $\mathcal{I} = (-\infty, \infty)$, then we obtain from (2.26) by taking the limit for $a \to -\infty$ for any conjoined basis (X, U) of (H) the estimate

$$m(-\infty,\infty) \geq \widehat{m}_\infty(-\infty,\infty). \tag{2.27}$$

Note that additional results about the singular Sturmian comparison theorems for completely controllable systems (H) and ($\hat{\text{H}}$) will be presented in Corollary 2.26 and in Subsection 2.3.3.

In the next sections we will show how the above estimates are generalized to possibly abnormal systems (H) and ($\hat{\text{H}}$) on bounded and/or unbounded intervals $\mathcal{I}$. Moreover, we will also see that several results regarding possibly uncontrollable linear Hamiltonian systems are new even for systems, which are completely controllable.

2.2. General Sturmian theory on compact interval

The Sturmian separation and comparison theorems on the compact interval $\mathcal{I}$ for possibly abnormal (or uncontrollable) systems (H) were derived in [60, Corollary 4.8] and [91, Theorems 1.2–1.5] by Kratz and Šimon Hilscher. This new theory employs the notion of a generalized (or proper) focal point of a conjoined basis of (H), which was introduced in [59] by Kratz and subsequently more specified [96] by Wahrheit, who defined the corresponding multiplicities of left and right proper focal points, see equations (2.30) and (2.31) below.

In this section we consider the compact interval $\mathcal{I} = [a, b]$. In the previous section we saw that the classical Sturmian theory for linear Hamiltonian system (H) is based on the complete controllability assumption. When this assumption is removed, Kratz and independently Fabbri, Johnson, and Núñez showed in [59, Theorem 3] and [42, Proof of Lemma 3.6(a)] the following result.

Proposition 2.2. *Assume that the Legendre condition (1.3) holds on $\mathcal{I} = [a, b]$. Then for any conjoined basis (X, U) of system(H) the kernel of $X(t)$ is piecewise constant on $[a, b]$. More precisely, for* *a given conjoined basis* (X,U) *of* (H) *there exists a partition* $a=t_0<t_1<\cdots<t_m=b$ *such that* $\operatorname{Ker} X(t)$ *is constant on the open interval* (t_j,t_{j+1}) *for all* $j\in\{0,1,\dots,m-1\}$ *and*

$$\operatorname{Ker} X(t_j^-)\subseteq \operatorname{Ker} X(t_j),\quad j\in\{1,2,\dots,m\}, \tag{2.28}$$

$$\operatorname{Ker} X(t_j^+)\subseteq \operatorname{Ker} X(t_j),\quad j\in\{0,1,\dots,m-1\}. \tag{2.29}$$

The quantities $\operatorname{Ker} X(t_j^{\pm})$ in (2.28) and (2.29) denote the limits of the constant set $\operatorname{Ker} X(t)$ as $t\to t_j^{\pm}$. The inclusions in (2.28) and (2.29) follow from the continuity of the matrix $X(t)$ on $[a,b]$. In the subsequent work [96], Wahrheit defined the point $t_0\in(a,b]$ to be a *left proper focal point* of (X,U) if $\operatorname{Ker} X(t_0^-)\subsetneqq \operatorname{Ker} X(t_0)$, with the multiplicity

$$m_L(t_0):=\operatorname{def} X(t_0)-\operatorname{def} X(t_0^-)=\operatorname{rank} X(t_0^-)-\operatorname{rank} X(t_0). \tag{2.30}$$

In a similar way we define $t_0\in[a,b)$ to be a *right proper focal point* of (X,U) by the condition $\operatorname{Ker} X(t_0^+)\subsetneqq \operatorname{Ker} X(t_0)$, with the multiplicity

$$m_R(t_0):=\operatorname{def} X(t_0)-\operatorname{def} X(t_0^+)=\operatorname{rank} X(t_0^+)-\operatorname{rank} X(t_0). \tag{2.31}$$

The notations $\operatorname{def} X(t_0^{\pm})$ and $\operatorname{rank} X(t_0^{\pm})$ represent the one-sided limits at t_0 of the piecewise constant quantities $\operatorname{def} X(t)$ and $\operatorname{rank} X(t)$.

Let (X,U) and $(\hat X,\hat U)$ be two conjoined bases of system (H). Moreover, given a point $s\in[a,b]$ let (X_s,U_s) be the principal solution of (H) at the point $s\in[a,b]$ as we defined in (1.12). We set

$$Y(t):=\begin{pmatrix}X(t)\\U(t)\end{pmatrix},\quad \hat Y(t):=\begin{pmatrix}\hat X(t)\\ \hat U(t)\end{pmatrix},\quad Y_s(t):=\begin{pmatrix}X_s(t)\\U_s(t)\end{pmatrix},\quad t\in[a,b], \tag{2.32}$$

which are $2n\times n$ matrix solutions of system (H). This notation will be used in particular when we deal with conjoined bases of system (H) combined together with the comparative index. It will be also useful for calculating the Wronskian of two conjoined bases, as it is shown in (1.38).

For convenience we denote by $m_L(a,b]$, $\widehat m_L(a,b]$, and $m_{Ls}(a,b]$ the total number of left proper focal points of Y, $\hat Y$, and Y_s in the half-open interval $(a,b]$, respectively. Similarly, we denote by $m_R[a,b)$, $\widehat m_R[a,b)$, and $m_{Rs}[a,b)$ the total number of right proper focal points of Y, $\hat Y$, and Y_s in the half-open interval $[a,b)$, respectively. We note that the left and right proper focal points are always counted including their multiplicities. By (2.30) and (2.31) we then have the equalities

$$m_L(a,b]=\sum_{t\in(a,b]} m_L(t),\quad m_R[a,b)=\sum_{t\in[a,b)} m_R(t). \tag{2.33}$$

Under (1.3) these sums are always finite, compare with the notation in (2.2).

The first Sturmian separation theorems for a possibly uncontrollable system (H) on bounded interval $\mathcal I$ were derived in [91] by Šimon Hilscher by using the eigenvalue theory for a certain perturbed linear Hamiltonian system. In [91, Theorem 1.4] and [60, Remark 4.7] it is shown that under (1.3) for any conjoined basis (X,U) of system (H) we have the estimates

$$m_{La}(a,b]\le m_L(a,b]\le m_{La}(a,b]+n, \tag{2.34}$$

$$m_{Rb}[a,b)\le m_R[a,b)\le m_{Rb}[a,b)+n. \tag{2.35}$$

These inequalities imply, see [91, Theorem 1.5], that for any conjoined bases (X,U) and $(\hat X,\hat U)$ of (H) we have the estimates

$$\big|m_L(a,b]-\widehat m_L(a,b]\big|\le n, \tag{2.36}$$

$$\big|m_R[a,b)-\widehat m_R[a,b)\big|\le n, \tag{2.37}$$

which generalize the results in (2.3) and (2.7) to possibly uncontrollable systems. Moreover, from [60, Corollary 4.8] we know that

$$m_{La}(a,b]=m_{Rb}[a,b). \tag{2.38}$$

The first Sturmian comparison theorems for two possibly uncontrollable systems (H) and ($\hat{\text{H}}$) satisfying the majorant condition (2.17) and the Legendre condition (2.18) were also derived in [91]. Note that the roles of the systems (H) and ($\hat{\text{H}}$) in [91] are interchanged. More precisely and with the notation in (2.17), the result in [91, Theorem 1.2] states that for any conjoined basis $(\hat{X}, \hat{U})$ of system ($\hat{\text{H}}$) we have the estimate

$$\widehat{m}_L(a,b] \leq m_{La}(a,b] + n, \tag{2.39}$$

while [91, Theorem 1.3] states that for any conjoined basis (X,U) of (H) we have the estimate

$$m_L(a,b] \geq \widehat{m}_{La}(a,b]. \tag{2.40}$$

By using [60, Remark 4.7] the estimates in (2.39) and (2.40) can be reformulated as

$$\widehat{m}_R[a,b) \leq m_{Rb}[a,b) + n, \tag{2.41}$$

$$m_R[a,b) \geq \widehat{m}_{Rb}[a,b). \tag{2.42}$$

We will comment about our contribution to this subject in Section 2.3.

2.2.1. Sturmian separation theorems. Using the above notation we can formulate a precise Sturmian separation theorem for system (H), which involves the comparative index when dealing with the left proper focal points and the dual comparative index when dealing with the right proper focal points. In this subsection we present the mains results from [84, Sections 4–6], see Appendix C. We also add some closely related results from [89, Section 1 and 3]. The next result was independently obtained also in [34, Theorem 2.3] by Elyseeva.

Theorem 2.3 (Sturmian separation theorem)**.** *Assume that* (1.3) *holds. Then for any conjoined bases* Y *and* $\hat{Y}$ *of* (H) *we have the equalities*

$$m_L(a,b] - \widehat{m}_L(a,b] = \mu\big(Y(b), \hat{Y}(b)\big) - \mu\big(Y(a), \hat{Y}(a)\big), \tag{2.43}$$

$$m_R[a,b) - \widehat{m}_R[a,b) = \mu^*\big(Y(a), \hat{Y}(a)\big) - \mu^*\big(Y(b), \hat{Y}(b)\big). \tag{2.44}$$

In the following we provide a formula, which relates the number of left proper focal points in $(a,b]$ and the number of right proper focal points in $[a,b)$ for one conjoined basis of (H). This extends the information provided in (2.5) to possibly uncontrollable system (H).

Theorem 2.4. *Assume that* (1.3) *holds. Then for any conjoined basis* (X,U) *of* (H) *its numbers of left proper focal points in* $(a,b]$ *and right proper focal points in* $[a,b)$ *satisfy*

$$m_L(a,b] + \operatorname{rank} X(b) = m_R[a,b) + \operatorname{rank} X(a). \tag{2.45}$$

The next two results demonstrate that the principal solutions Y_a and Y_b of system (H) at the points a and b play a prominent roles in the presented Sturmian theory of (H). The numbers

$$m_{La}(a,b], \quad m_{Ra}[a,b), \quad m_{Lb}(a,b], \quad m_{Rb}[a,b) \tag{2.46}$$

turn out to be essential parameters of system (H) on the interval $[a,b]$. More precisely, they are optimal bounds for the numbers of left and right proper focal points of any conjoined basis (X,U) in $(a,b]$ and $[a,b)$, respectively. In the next results we use the notation from (2.32) regarding the principal solutions Y_a and Y_b of (H).

Theorem 2.5. *Assume that* (1.3) *holds. With the notation in* (2.46) *we have for the left and right focal points of the principal solutions* (X_a,U_a) *and* (X_b,U_b) *in* $(a,b]$ *and* $[a,b)$*, respectively, the equalities*

$$m_{Lb}(a,b] = m_{La}(a,b] + \operatorname{rank} X_a(b) = m_{La}(a,b] + \operatorname{rank} X_b(a), \tag{2.47}$$

$$m_{Ra}[a,b) = m_{Rb}[a,b) + \operatorname{rank} X_b(a) = m_{Rb}[a,b) + \operatorname{rank} X_a(b), \tag{2.48}$$

$$m_{Ra}[a,b) = m_{Lb}(a,b], \quad m_{La}(a,b] = m_{Rb}[a,b). \tag{2.49}$$

Note that the second equality in (2.49) is known in (2.38), while the first equality in (2.49) is new.

Theorem 2.6 (Sturmian separation theorem)**.** *Assume that* (1.3) *holds. Then for any conjoined basis* (X,U) *of* (H) *we have the inequalities*

$$m_{La}(a,b] \leq m_L(a,b] \leq m_{Lb}(a,b], \tag{2.50}$$

$$m_{Rb}[a,b) \leq m_R[a,b) \leq m_{Ra}[a,b). \tag{2.51}$$

According to (2.49) in Theorem 2.5, the lower bounds in (2.50) and (2.51) are the same, as well as the upper bounds in (2.50) and (2.51) are the same. Moreover, these lower and upper bounds are independent on the conjoined basis (X, U). Since these bounds are attained for the specific choices of $(X, U) := (X_a, U_a)$ and $(X, U) := (X_b, U_b)$, the inequalities in (2.50) and (2.51) cannot be improved – in the sense that the estimates (2.50) and (2.51) are satisfied *for all* conjoined bases (X, U) of (H).

In the context of Theorem 2.3 the principal solutions Y_a and Y_b can be viewed as reference solutions of system (H) when counting the number of left and right proper focal points in $(a, b]$ and $[a, b)$, respectively. More precisely, every conjoined basis Y of system (H) satisfies the exact formulas

$$m_L(a, b] = m_{La}(a, b] + \mu(Y(b), Y_a(b)), \tag{2.52}$$

$$m_R[a, b) = m_{Rb}[a, b) + \mu^*(Y(a), Y_b(a)). \tag{2.53}$$

Inequalities (2.50)–(2.51) together with (2.47)–(2.48) imply directly the optimal and the universal bound for the difference between the numbers of proper focal points of any two conjoined bases of system (H). This result improves the estimates in (2.36) and (2.37). Note that this result is also new in the controllable case, where it generalizes the estimate presented in (2.3).

Corollary 2.7 (Sturmian separation theorem)**.** *Assume that* (1.3) *holds. Then for any conjoined bases* (X, U) *and* $(\hat{X}, \hat{U})$ *of* (H) *we have the estimates*

$$\big|\, m_L(a, b] - \widehat{m}_L(a, b]\, \big| \le \operatorname{rank} X_a(b) = \operatorname{rank} X_b(a) \le n, \tag{2.54}$$

$$\big|\, m_R[a, b) - \widehat{m}_R[a, b)\, \big| \le \operatorname{rank} X_b(a) = \operatorname{rank} X_a(b) \le n, \tag{2.55}$$

$$\big|\, m_L(a, b] - \widehat{m}_R[a, b)\, \big| \le \operatorname{rank} X_a(b) = \operatorname{rank} X_b(a) \le n. \tag{2.56}$$

The results in Theorem 2.6 also pose the natural question, whether for any given integers ℓ and r within the lower and upper bounds in (2.50) and (2.51) there exists a conjoined basis Y of system (H), for which the equalities

$$m_L(a, b] = \ell \quad \text{and} \quad m_R[a, b) = r \tag{2.57}$$

hold. And if so, then how to determine such a conjoined basis. The answers to both these questions are presented in the next statement from [89, Theorem 1.1].

Theorem 2.8. *Assume that* (1.3) *holds. Then for any integers* ℓ *and* r *satisfying*

$$m_{La}(a, b] \le \ell \le m_{Lb}(a, b] \quad \textit{and} \quad m_{Rb}[a, b) \le r \le m_{Ra}[a, b) \tag{2.58}$$

there exists a conjoined basis Y *of* (H) *such that* (2.57) *holds. Moreover, if* $\ell \ge r$*, then the conjoined basis* Y *can be chosen with* $X(a) = I$*, and if* $\ell \le r$*, then the conjoined basis* Y *can be chosen with* $X(b) = I$*. In particular, when* $\ell = r$ *the conjoined basis* Y *may be chosen with both* $X(a)$ *and* $X(b)$ *invertible.*

In the case when system (H) is completely controllable on $[a, b]$, then Theorem 2.8 represents a generalization of the classical result by Reid, see e.g. [69, Theorem V.6.3, pg. 284–285] or [67, Theorem VII.5.1] in combination with [59, Theorem 1]. We display this result explicitly for an easy comparison with the results in Section 1.3 regarding the explicit Riccati equation (R), see [89, Theorem 3.2 and Remark 3.3] for more details.

Theorem 2.9. *Assume that* (1.3) *holds and system* (H) *is completely controllable on* $[a, b]$*. Then the following statements are equivalent.*

- (i) *There exists a conjoined basis* (X, U) *of* (H) *such that* $X(t)$ *is invertible on* $(a, b]$*.*
- (ii) *For any integer* r *with* $0 \le r \le n$ *there exists a conjoined basis* (X, U) *of* (H) *such that* $X(t)$ *is invertible on* $(a, b]$ *and* $m(a) = r$*.*
- (iii) *There exists a conjoined basis* (X, U) *of* (H) *such that* $X(t)$ *is invertible on* $[a, b)$*.*
- (iv) *For any integer* ℓ *with* $0 \le \ell \le n$ *there exists a conjoined basis* (X, U) *of* (H) *such that* $X(t)$ *is invertible on* $[a, b)$ *and* $m(b) = \ell$*.*

The results in Theorem 2.9 are important for applications. For example, in the theory of Riccati matrix differential equations, see e.g. [58, 67, 68, 77], they provide a sufficient and also a necessary condition for the existence of a symmetric solution of (R) on the whole intervals $[a,b]$, $(a,b]$, or $[a,b)$ by considering the Riccati quotient $Q(t) = U(t)\,X^{-1}(t)$ in (1.25). We remark that the conjoined basis (X,U) in part (ii) of Theorem 2.9 can be constructed by prescribing the initial conditions at the point b. More precisely, all such conjoined bases (X,U) (up to a constant right nonsingular multiple) are determined as

$$X(b) = I, \quad U(b) = D + Q_a(b), \quad D \leq 0, \quad \operatorname{ind} D = \operatorname{rank} D = n - r, \tag{2.59}$$

where Q_a is the Riccati quotient in (1.26) associated with the principal solution Y_a. Similarly, all conjoined bases Y (up to a constant right nonsingular multiple) in part (iv) of Theorem 2.9 are constructed by the initial conditions at the point a. Namely, we have

$$X(a) = I, \quad U(a) = D + Q_b(a), \quad D \geq 0, \quad \operatorname{ind}(-D) = \operatorname{rank} D = n - \ell, \tag{2.60}$$

where Q_b is the Riccati quotient in (1.26) associated with the principal solution Y_b.

The above results from the Sturmian theory of system (H) allow to derive additional properties of the comparative index involving two conjoined bases of (H). These properties are of a local character and they are expressed in terms of the limit of the comparative index, see [84, Theorems 6.1 and 6.3] and [34, Theorem 2.3].

Theorem 2.10. *Assume that (1.3) holds. Then for any two conjoined bases Y and $\hat{Y}$ of system (H) the following properties are satisfied.*

(i) *The comparative index $\mu\big(Y(t), \hat{Y}(t)\big)$ is piecewise constant on $[a,b]$ and right continuous on $[a,b)$. In addition,*

$$\lim_{t\to t_0^-} \mu\big(Y(t), \hat{Y}(t)\big) = \mu\big(Y(t_0), \hat{Y}(t_0)\big) - m_L(t_0) + \widehat{m}_L(t_0), \quad t_0 \in (a,b], \tag{2.61}$$

and $\mu\big(Y(t), \hat{Y}(t)\big)$ is not left continuous at $t_0 \in (a,b]$ if and only if $m_L(t_0) \neq \widehat{m}_L(t_0)$.

(ii) *The dual comparative index $\mu^*\big(Y(t), \hat{Y}(t)\big)$ is piecewise constant on $[a,b]$ and left continuous on $(a,b]$. In addition,*

$$\lim_{t\to t_0^+} \mu^*\big(Y(t), \hat{Y}(t)\big) = \mu^*\big(Y(t_0), \hat{Y}(t_0)\big) - m_R(t_0) + \widehat{m}_R(t_0), \quad t_0 \in [a,b), \tag{2.62}$$

and $\mu^*\big(Y(t), \hat{Y}(t)\big)$ is not right continuous at $t_0 \in [a,b)$ if and only if $m_R(t_0) \neq \widehat{m}_R(t_0)$.

The next result shows how to compute the multiplicities in (2.30) and (2.31) of left and right proper focal points of Y at some point t_0 by a limit involving the comparative index.

Theorem 2.11. *Assume that (1.3) holds. Let Y be a conjoined basis of (H) and let Y_t be the principal solution of (H) at the point t in (2.32). Then we have*

$$m_L(t_0) = \lim_{t\to t_0^-} \mu\big(Y(t_0), Y_t\,(t_0)\big), \quad t_0 \in (a,b], \tag{2.63}$$

$$m_R(t_0) = \lim_{t\to t_0^+} \mu^*\big(Y(t_0), Y_t\,(t_0)\big), \quad t_0 \in [a,b). \tag{2.64}$$

2.2.2. Sturmian comparison theorems. In this is subsection we comment on the Sturmian separation theorems (in particular on the results in Theorems 2.3 and 2.10) from a more general context. More precisely, we present the Sturmian comparison theorem for conjoined bases of two systems of the form (H) and ($\hat{\text{H}}$) satisfying the majorant condition (2.17). Along with the basic systems (H) and ($\hat{\text{H}}$) we will also consider a certain transformed linear Hamiltonian system

$$\tilde{y}' = \mathcal{J}\tilde{\mathcal{H}}(t)\,\tilde{y}, \quad t \in \mathcal{I}, \tag{$\tilde{\text{H}}$}$$

which is related to (H) and ($\hat{\text{H}}$) by a symplectic transformation, see formula (2.70) below.

The following exact formula for expressing the numbers of left and right proper focal points of conjoined bases Y and $\hat{Y}$ of systems (H) and ($\hat{\text{H}}$) on $\mathcal{I} = [a,b]$ was derived in [34, Theorem 2.2] by

Elyseeva. In the spirit of (2.33), we denote by $\widehat{m}_L(a,b]$ and $\widehat{m}_R[a,b)$ the total number of left and right proper focal points of the conjoined basis $\hat{Y}$ of system ($\hat{\mathrm{H}}$) in the indicated interval. Similar notation $\widetilde{m}_L(a,b]$ and $\widetilde{m}_R[a,b)$ will be used for the conjoined basis $\tilde{Y}$ of system ($\tilde{\mathrm{H}}$), see below. Under (1.3) these sums are always finite. For convenience we define the constant $2n\times n$ matrix

$$E := (0,\ I)^T, \tag{2.65}$$

which can be considered, in view of (1.12), as the initial condition for the principal solutions Y_s, $\hat{Y}_s$, $\tilde{Y}_s$ of systems (H), ($\hat{\mathrm{H}}$), ($\tilde{\mathrm{H}}$) at the point $s\in\mathcal{I}$.

Theorem 2.12. *Assume that* (2.17) *and* (2.18) *hold on* $\mathcal{I}=[a,b]$ *and let* Y *and* $\hat{Y}$ *be any conjoined bases of* (H) *and* ($\hat{\mathrm{H}}$). *Let* $\hat{Z}$ *be a fundamental matrix of* ($\hat{\mathrm{H}}$) *satisfying* $\hat{Y}(t)=\hat{Z}(t)\,E$ *on* $[a,b]$, *where the matrix* E *is given in* (2.65), *and consider the function* $\tilde{Y}(t):=\hat{Z}^{-1}(t)\,Y(t)$ *on* $[a,b]$. *Then the comparative index* $\mu(Y(t),\hat{Y}(t))$ *is piecewise constant on* $[a,b]$ *and right-continuous on* $[a,b)$ *and for every* $t_0\in(a,b]$ *the multiplicities* $m_L(t_0)$, $\widehat{m}_L(t_0)$, *and* $\widetilde{m}_L(t_0)$ *of left proper focal points of* Y, $\hat{Y}$, *and* $\tilde{Y}$ *at* t_0 *defined through* (2.30) *satisfy the equality*

$$m_L(t_0)-\widehat{m}_L(t_0)=\widetilde{m}_L(t_0)+\mu(Y(t_0),\hat{Y}(t_0))-\lim_{t\to t_0^-}\mu(Y(t),\hat{Y}(t)). \tag{2.66}$$

Moreover, the numbers of left proper focal points of Y *and* $\hat{Y}$ *in* $(a,b]$ *are connected by*

$$m_L(a,b]-\widehat{m}_L(a,b]=\widetilde{m}_L(a,b]+\mu(Y(b),\hat{Y}(b))-\mu(Y(a),\hat{Y}(a)), \tag{2.67}$$

where $\widetilde{m}_L(a,b]$ *is the number of left proper focal points in* $(a,b]$ *of the auxiliary function* $\tilde{Y}$.

A corresponding result for the right proper focal points in $[a,b)$ can be derived by an analogous method to the proof of Theorem 2.12 in [34]. Alternatively, we may use the relationship in (2.45) in Theorem 2.4 between the left and right proper focal points of Y.

Theorem 2.13. *Under the assumptions of Theorem 2.12, for any conjoined bases* Y *and* $\hat{Y}$ *of* (H) *and* ($\hat{\mathrm{H}}$) *the dual comparative index* $\mu^*(Y(t),\hat{Y}(t))$ *is piecewise constant on* $[a,b]$ *and left-continuous on* $(a,b]$ *and for every* $t_0\in[a,b)$ *the multiplicities* $m_R(t_0)$, $\widehat{m}_R(t_0)$, *and* $\widetilde{m}_R(t_0)$ *of right proper focal points of* Y, $\hat{Y}$, *and* $\tilde{Y}:=\hat{Z}^{-1}Y$ *at* t_0 *defined through* (2.31) *satisfy the equality*

$$m_R(t_0)-\widehat{m}_R(t_0)=\widetilde{m}_R(t_0)+\mu^*(Y(t_0),\hat{Y}(t_0))-\lim_{t\to t_0^+}\mu^*(Y(t),\hat{Y}(t)). \tag{2.68}$$

Moreover, the numbers of right proper focal points of Y *and* $\hat{Y}$ *in* $[a,b)$ *are connected by*

$$m_R[a,b)-\widehat{m}_R[a,b)=\widetilde{m}_R[a,b)+\mu^*(Y(a),\hat{Y}(a))-\mu^*(Y(b),\hat{Y}(b)), \tag{2.69}$$

where $\widetilde{m}_R[a,b)$ *is the number of right proper focal points in* $[a,b)$ *of the auxiliary function* $\tilde{Y}$.

We note that the symplectic fundamental matrix $\hat{Z}$ of ($\hat{\mathrm{H}}$) in Theorems 2.12 and 2.13 has the form $\hat{Z}=(*\,,\hat{Y})$. Moreover, it is easy to verify (see [26]) that the function $\tilde{Y}:=\hat{Z}^{-1}Y$ is a conjoined basis of the transformed linear Hamiltonian system ($\tilde{\mathrm{H}}$), whose coefficient matrix

$$\tilde{\mathcal{H}}(t):=\hat{Z}^T(t)\,[\mathcal{H}(t)-\hat{\mathcal{H}}(t)]\,\hat{Z}(t),\quad t\in\mathcal{I}, \tag{2.70}$$

satisfies $\tilde{\mathcal{H}}(t)\geq 0$ on $\mathcal{I}$ under (2.17). In particular, the Legendre condition

$$\tilde{B}(t)\geq 0\quad \text{for all } t\in\mathcal{I} \tag{2.71}$$

holds, where $\tilde{B}(t)$ is the lower right $n\times n$ block of the matrix $\tilde{\mathcal{H}}(t)$. Condition (2.71) implies that the quantities $\widetilde{m}_L(t_0)$, $\widetilde{m}_R(t_0)$ in (2.66), (2.68) and the quantities $\widetilde{m}_L(a,b]$, $\widetilde{m}_R[a,b)$ in (2.67), (2.69) are correctly defined.

When the two systems (H) and ($\hat{\mathrm{H}}$) coincide, i.e., when $\hat{\mathcal{H}}(t)\equiv\mathcal{H}(t)$ on $\mathcal{I}$, then $\tilde{\mathcal{H}}(t)\equiv 0$ on $\mathcal{I}$ by (2.70) and hence, all conjoined bases $\tilde{Y}$ of ($\tilde{\mathrm{H}}$) are constant on $\mathcal{I}$ and we have $\widetilde{m}_L(a,b]=0=\widetilde{m}_R[a,b)$. In this case the results in (2.67) and (2.69) reduce to formulas (2.43) and (2.44) in Theorem 2.3. Similarly, the results in (2.66) and (2.68) reduce to formulas (2.61) and (2.62) in Theorem 2.10, since in this case we have $\widetilde{m}_L(t_0)=0=\widetilde{m}_R(t_0)$.

2.3. General Singular Sturmian theory on unbounded intervals

In this section we present our fundamental contributions to the singular Sturmian theory for nonoscillatory and possibly uncontrollable linear Hamiltonian systems (H) on the unbounded interval $\mathcal{I} = [a, \infty)$. These new results employ two key tools, namely,

- the theory of minimal principal and maximal antiprincipal solutions of (H) at infinity, and
- the concept of a multiplicity of a focal point at infinity.

In [79, 81, 85] we showed that every conjoined basis Y of (H) with constant kernel on $[\alpha, \infty)$ with $d[\alpha, \infty) = d_\infty$ satisfies

$$n - d_\infty \le \operatorname{rank} X(t) \le n, \quad t \in [\alpha, \infty), \tag{2.72}$$

$$0 \le \operatorname{rank} T_{\alpha,\infty} \le n - d_\infty, \quad T_{\alpha,\infty} := \lim_{t\to\infty} \left(\int_\alpha^t X^\dagger(s)\, B(s)\, X^{\dagger T}(s)\, \mathrm{d}s \right)^\dagger. \tag{2.73}$$

We recall from Section 1.4 that Y is the minimal principal solution at infinity if the corresponding matrix in (2.73) satisfies $T_{\alpha,\infty} = 0$ and if $\operatorname{rank} X(t) = n - d_\infty$ on $[\alpha, \infty)$. Moreover, according to [81, Definition 5.1], a conjoined basis Y of (H) with constant kernel on $[\alpha, \infty)$ with $d[\alpha, \infty) = d_\infty$ is a *maximal antiprincipal solution at infinity* if eventually $\operatorname{rank} X(t) = n$ and the corresponding matrix $T_{\alpha,\infty}$ in (2.73) satisfies

$$\operatorname{rank} T_{\alpha,\infty} = n - d_\infty,$$

i.e., the rank of the matrix $T_{\alpha,\infty}$ is maximal according to (2.73). In this section we will use the notation Y_∞ for the minimal principal solution at infinity (in Section 1.4 we used the notation $(\hat{X}_{\min}, \hat{U}_{\min})$), while for a maximal antiprincipal solution at infinity we will use the notation $\bar{Y}_\infty$. By [87, Proposition 2.3] we know that a given maximal antiprincipal solution $\bar{Y}_\infty$ completes the minimal principal solution Y_∞ (or its suitable invertible multiple) to a symplectic fundamental matrix Z_∞ of system (H). That is, we have

$$Z_\infty(t) = \begin{pmatrix} \bar{Y}_\infty(t) & Y_\infty(t) \end{pmatrix}, \quad t \in [a, \infty), \quad W(\bar{Y}_\infty, Y_\infty) = I, \tag{2.74}$$

where $W(\bar{Y}_\infty, Y_\infty)$ is the Wronskian of the conjoined bases $\bar{Y}_\infty$ and Y_∞. Then every conjoined basis Y of (H) can be uniquely represented by a constant $2n \times n$ matrix C_∞ satisfying

$$Y(t) = Z_\infty(t)\, C_\infty, \quad t \in [a, \infty), \quad C_\infty := \begin{pmatrix} -W(Y_\infty, Y) \\ W(\bar{Y}_\infty, Y) \end{pmatrix}. \tag{2.75}$$

The following notion appeared in [86] and it is completely new in the theory of linear Hamiltonian differential systems. It provides a unified view on the principal solutions of system (H) at a finite point and at infinity, see equality (2.77) in Theorem 2.15 below.

Definition 2.14 (Multiplicity of focal point at infinity)**.** Let Y be a conjoined basis of system (H) with constant kernel on the interval $[\alpha, \infty)$ for some $\alpha \in [\alpha_\infty, \infty)$ with α_∞ defined in (1.7). We say that Y has a *(left) proper focal point at infinity* if $d_\infty + \operatorname{rank} T_{\alpha,\infty} < n$ with the multiplicity

$$m_L(\infty) := n - d_\infty - \operatorname{rank} T_{\alpha,\infty}, \tag{2.76}$$

where d_∞ is the maximal order of abnormality of (H) in (1.6) and $T_{\alpha,\infty}$ is the matrix defined in (2.73) corresponding to Y.

In accordance with (2.73) we note that under (1.3) the number $m_L(\infty)$ defined in (2.76) is always nonnegative. Moreover, it does not depend on the particular choice of the point $\alpha \in [\alpha_\infty, \infty)$, for which the conjoined basis Y has constant kernel on $[\alpha, \infty)$. In particular, we have the estimates $0 \le m_L(\infty) \le n - d_\infty$. It follows that the conjoined basis Y has no focal point at infinity, i.e. $m_L(\infty) = 0$, if and only if Y is an antiprincipal solution of (H) at infinity. Similarly, the multiplicity $m_L(\infty) = n - d_\infty$ is maximal possible if and only if Y is a principal solution of (H) at infinity.

The next result, see [86, Theorem 3.3], provides a way for computing the multiplicity of the focal point at infinity in terms of the rank of the genus of a conjoined basis Y and the rank of the Wronskian of Y with the minimal principal solution Y_∞ at infinity.

Theorem 2.15. *Assume that* (1.3) *holds with* $[a,\infty)$ *and system* (H) *is nonoscillatory. Let* Y *be a conjoined basis of* (H) *belonging to a genus* $\mathcal{G}$. *Then the multiplicity of the focal point of* Y *at infinity defined in* (2.76) *satisfies the formula*

$$m_L(\infty) = \operatorname{rank}\mathcal{G} - \operatorname{rank} W(Y_\infty, Y), \tag{2.77}$$

where the quantity $\operatorname{rank}\mathcal{G}$ *is defined in* (1.10) *and* $W(Y_\infty, Y)$ *is the Wronskian of* Y_∞ *and* Y.

Remark 2.16. When system (H) corresponds to the second order Sturm–Liouville differential equation, then the statement of Theorem 2.15 characterizes the principal solutions at infinity as those solutions with $m_L(\infty) = 1$. On the other hand, all nonprincipal solutions at infinity satisfy $m_L(\infty) = 0$.

2.3.1. Singular Sturmian separation theorems. In this subsection we present the main results from [86, Sections 5–7], see Appendix D. More precisely, we provide Sturmian separation theorems for conjoined bases of a nonoscillatory system (H) on the unbounded intervals $(a,\infty]$, resp. $[a,\infty)$. We emphasize that the results regarding left proper focal points include the multiplicity of proper focal point at infinity, which was introduced in Definition 2.14. In addition, we will also provide the corresponding results for the open interval (a,∞). We note that the results in this subsection hold also with the left endpoint $a=-\infty$ for the intervals $\mathcal{I}=(-\infty,b]$ or $\mathcal{I}=(-\infty,\infty)$, if we consider the corresponding concept of the minimal principal solution of (H) at minus infinity. A detailed analysis of this case is presented in [86, Remarks 5.16 and 8.1], see Appendix D.

The following result corresponds to formulas (2.43) and (2.44), where a compact interval $\mathcal{I}=[a,b]$ was considered. It also justifies the necessity of including the multiplicity of proper focal point at infinity in the singular Sturmian theory of system (H).

Theorem 2.17 (Singular Sturmian separation theorem)**.** *Assume that* (1.3) *holds with* $\mathcal{I}=[a,\infty)$ *and system* (H) *is nonoscillatory. Then for any conjoined bases* Y *and* $\hat{Y}$ *of* (H) *we have the equalities*

$$m_L(a,\infty] - \widehat{m}_L(a,\infty] = \mu\big(C_\infty, \hat{C}_\infty\big) - \mu\big(Y(a), \hat{Y}(a)\big), \tag{2.78}$$

$$m_R[a,\infty) - \widehat{m}_R[a,\infty) = \mu^*\big(Y(a), \hat{Y}(a)\big) - \mu^*\big(C_\infty, \hat{C}_\infty\big), \tag{2.79}$$

where C_∞ *and* $\hat{C}_\infty$ *are the constant matrices in* (2.75) *corresponding to* Y *and* $\hat{Y}$.

By considering a special choice of the conjoined basis $\hat{Y}$ in Theorem 2.17 we obtain formulas for the exact numbers of left and right proper focal points of any conjoined basis Y of (H) in the intervals $(a,\infty]$ and $[a,\infty)$. They highlight the importance of the minimal principal solution Y_∞ of (H) at infinity and the principal solution Y_a of (H) at a in counting the numbers $m_L(a,\infty]$ and $m_R[a,\infty)$. They correspond to formulas (2.52) and (2.53), where a compact interval $\mathcal{I}=[a,b]$ was considered. For convenience we denote by $m_{L\infty}(a,\infty]$ the total number of left proper focal points of the minimal principal solution Y_∞ of (H) at infinity in the interval $(a,\infty]$. Similarly, we denote by $m_{R\infty}[a,\infty)$ the total number of right proper focal points of Y_∞ in the interval $[a,\infty)$.

Theorem 2.18. *Assume that* (1.3) *holds with* $\mathcal{I}=[a,\infty)$ *and system* (H) *is nonoscillatory. Then for any conjoined basis* Y *of* (H) *we have the equalities*

$$m_L(a,\infty] = m_{La}(a,\infty] + \mu\big(C_\infty, C_\infty^a\big), \tag{2.80}$$

$$m_R[a,\infty) = m_{R\infty}[a,\infty) + \mu^*\big(Y(a), Y_\infty(a)\big), \tag{2.81}$$

where C_∞ *and* C_∞^a *are the constant matrices in* (2.75) *corresponding to* Y *and* Y_a.

In the next statement we connect the multiplicities of left and right proper focal points of one conjoined basis Y of (H) in an unbounded interval. This result corresponds to formula (2.45).

Theorem 2.19. *Assume that* (1.3) *holds with* $\mathcal{I}=[a,\infty)$ *and system* (H) *is nonoscillatory. Let* Y_∞ *be the minimal principal solution of* (H) *at infinity. Then for any conjoined basis* Y *of* (H) *its numbers of left proper focal points in the interval* $(a,\infty]$ *and right proper focal points in the interval* $[a,\infty)$ *satisfy the equality*

$$m_L(a,\infty] + \operatorname{rank} W(Y_\infty, Y) = m_R[a,\infty) + \operatorname{rank} X(a). \tag{2.82}$$

In the remaining results of this subsection we will use the principal solution Y_a of (H) at the point a and the minimal principal solution Y_∞ of (H) at infinity as important ingredients in the presented results from the singular Sturmian theory on the unbounded interval $\mathcal{I} = [a, \infty)$. The following statement relates the numbers of left and right proper focal points of Y_a and Y_∞ in $(a, \infty]$ and $[a, \infty)$. This result corresponds to formulas (2.47)–(2.49) in Theorem 2.5.

Theorem 2.20. *Assume that the Legendre condition (1.3) holds with $\mathcal{I} = [a, \infty)$ and system (H) is nonoscillatory. Then we have the formulas*

$$m_{L\infty}(a,\infty] = m_{La}(a,\infty] + \operatorname{rank} X_\infty(a), \quad m_{Ra}[a,\infty) = m_{R\infty}[a,\infty) + \operatorname{rank} X_\infty(a), \tag{2.83}$$

$$m_{La}(a,\infty] = m_{R\infty}[a,\infty), \quad m_{Ra}[a,\infty) = m_{L\infty}(a,\infty]. \tag{2.84}$$

We remark that equations (2.80) and (2.81) yield the lower bounds

$$m_L(a,\infty] \geq m_{La}(a,\infty], \quad m_R[a,\infty) \geq m_{R\infty}[a,\infty) \tag{2.85}$$

for the numbers of left and right proper focal points of any conjoined basis Y of (H) in the interval $(a, \infty]$ and $[a, \infty)$. These two lower bounds are the same according to (2.84). Moreover, the second estimate in (2.85) generalizes the result of Došlý and Kratz in (2.14) to possibly uncontrollable system (H). In the next statement we provide the corresponding optimal upper bounds for the numbers $m_L(a, \infty]$ and $m_R[a, \infty)$. These estimates correspond to (2.50)–(2.51) in Theorem 2.6.

Theorem 2.21 (Singular Sturmian separation theorem)**.** *Assume that (1.3) holds with $\mathcal{I} = [a, \infty)$ and system (H) is nonoscillatory. Then for any conjoined basis Y of (H) we have*

$$m_{La}(a,\infty] \leq m_L(a,\infty] \leq m_{L\infty}(a,\infty], \tag{2.86}$$

$$m_{R\infty}[a,\infty) \leq m_R[a,\infty) \leq m_{Ra}[a,\infty). \tag{2.87}$$

The results in the above theorem yield the following optimal estimates for the difference of the numbers of the left and right proper focal points of any two conjoined bases of (H). They correspond to (2.54)–(2.56) in Corollary 2.7.

Corollary 2.22 (Singular Sturmian separation theorem)**.** *Assume that (1.3) holds with $\mathcal{I} = [a, \infty)$ and system (H) is nonoscillatory. Then for any conjoined bases Y and $\hat{Y}$ of (H) we have*

$$\big|\, m_L(a,\infty] - \widehat{m}_L(a,\infty] \,\big| \leq \operatorname{rank} X_\infty(a) \leq n,$$

$$\big|\, m_R[a,\infty) - \widehat{m}_R[a,\infty) \,\big| \leq \operatorname{rank} X_\infty(a) \leq n,$$

$$\big|\, m_L(a,\infty] - \widehat{m}_R[a,\infty) \,\big| \leq \operatorname{rank} X_\infty(a) \leq n.$$

In the next part of this subsection we analyze the numbers $m_L(a, \infty)$ and $m_R(a, \infty)$ of left and right proper focal points of a conjoined basis Y of (H) in the open interval (a, ∞). The motivation comes from possible practical applications, where the independent variable is always finite.

In the following statement we provide optimal lower and upper bounds for the numbers left and right proper focal points of any conjoined basis Y of (H) in the open interval (a, ∞). It is surprising that the optimal upper bounds for $m_L(a, \infty)$ and $m_R(a, \infty)$ are the same as in (2.86) and (2.87), i.e., they are equal to $m_{L\infty}(a, \infty]$ and $m_{Ra}[a, \infty)$, respectively.

Theorem 2.23. *Assume that (1.3) holds with $\mathcal{I} = [a, \infty)$ and system (H) is nonoscillatory. Then for any conjoined basis Y of (H) we have*

$$m_{La}(a,\infty) \leq m_L(a,\infty) \leq m_{L\infty}(a,\infty], \tag{2.88}$$

$$m_{R\infty}(a,\infty) \leq m_R(a,\infty) \leq m_{Ra}[a,\infty). \tag{2.89}$$

As an analogy of Corollary 2.22 we obtain from Theorem 2.23 the following estimates.

Corollary 2.24. *Assume that (1.3) holds with $\mathcal{I} = [a, \infty)$ and system (H) is nonoscillatory. Then for any two conjoined bases Y and $\hat{Y}$ of (H) we have*

$$\big|\, m_L(a,\infty) - \widehat{m}_L(a,\infty) \,\big| \leq \operatorname{rank} \mathcal{G}_a \leq n, \tag{2.90}$$

$$\big|\, m_R(a,\infty) - \widehat{m}_R(a,\infty) \,\big| \leq \operatorname{rank} X_\infty(a^+) \leq n, \tag{2.91}$$

where $\mathcal{G}_a$ is the genus of conjoined bases of (H)*, which contains the principal solution Y_a.*

The next result connects the multiplicities of left and right proper focal points of the principal solutions Y_a and Y_∞ in the open interval (a, ∞).

Theorem 2.25. *Assume that* (1.3) *holds with $\mathcal{I} = [a, \infty)$ and system* (H) *is nonoscillatory. Then we have the formula*

$$m_{La}(a, \infty) + \operatorname{rank} \mathcal{G}_a = m_{R\infty}(a, \infty) + \operatorname{rank} X_\infty(a^+), \tag{2.92}$$

where $\mathcal{G}_a$ is the genus of the principal solution Y_a. In particular, the equality

$$m_{La}(a, \infty) = m_{R\infty}(a, \infty) \tag{2.93}$$

holds if and only if $\operatorname{rank} \mathcal{G}_a = \operatorname{rank} X_\infty(a^+)$.

When system (H) is completely controllable then every conjoined basis Y of (H) has the matrix $X(t)$ invertible near a. In addition, if (H) is nonoscillatory, then $X(t)$ is also invertible near ∞. Therefore, in this case the condition $\operatorname{rank} \mathcal{G}_a = \operatorname{rank} X_\infty(a^+) = n$ is automatically satisfied and we get from Corollary 2.25 the following. This result is also new even in this special setting.

Corollary 2.26. *Assume that* (1.3) *holds with $\mathcal{I} = [a, \infty)$ and system* (H) *is completely controllable on $[a, \infty)$ and nonoscillatory. Then the principal solutions Y_a and Y_∞ have the same number of focal points in the open interval (a, ∞), i.e.,*

$$m_a(a, \infty) = m_\infty(a, \infty). \tag{2.94}$$

The above results from the singular Sturmian theory of system (H) on an unbounded interval allow to describe asymptotic properties of the comparative indices $\mu\big(Y(t), \hat{Y}(t)\big)$ and $\mu^*\big(Y(t), \hat{Y}(t)\big)$ when $t \to \infty$ for a pair of conjoined bases Y and $\hat{Y}$ of (H), see [86, Theorems 6.1 and 6.4] or Appendix D. This result corresponds to Theorem 2.10.

Theorem 2.27. *Assume that* (1.3) *holds with $\mathcal{I} = [a, \infty)$ and system* (H) *is nonoscillatory. Then for any two conjoined bases Y and $\hat{Y}$ of* (H) *the limits of the comparative indices $\mu\big(Y(t), \hat{Y}(t)\big)$ and $\mu^*\big(Y(t), \hat{Y}(t)\big)$ for $t \to \infty$ exist and*

$$\lim_{t\to\infty} \mu\big(Y(t), \hat{Y}(t)\big) = \mu\big(C_\infty, \hat{C}_\infty\big) - m_L(\infty) + \widehat{m}_L(\infty), \tag{2.95}$$

$$\lim_{t\to\infty} \mu^*\big(Y(t), \hat{Y}(t)\big) = \mu^*(C_\infty, \hat{C}_\infty), \tag{2.96}$$

where C_∞ and $\hat{C}_\infty$ are the constant matrices in (2.75) *corresponding to Y and $\hat{Y}$.*

In the next theorem we present a formula for calculating the multiplicity of the focal point at infinity of a conjoined basis Y of system (H) in terms of the comparative index of Y with the principal solution Y_t, respectively in terms of their representing matrices C_∞ and C_∞^t in (2.75). In addition, we provide an interesting representation of the maximal order of abnormality d_∞ of (H) in terms of the dual comparative index $\mu^*\big(C_\infty, C_\infty^t\big)$. This result corresponds to Theorem 2.11.

Theorem 2.28. *Assume that* (1.3) *holds with $\mathcal{I} = [a, \infty)$ and system* (H) *is nonoscillatory. Then for any conjoined basis Y of* (H) *the limits of the comparative indices $\mu\big(C_\infty, C_\infty^t\big)$ and $\mu^*\big(C_\infty, C_\infty^t\big)$ for $t \to \infty$ exist and*

$$\lim_{t\to\infty} \mu\big(C_\infty, C_\infty^t\big) = m_L(\infty), \tag{2.97}$$

$$\lim_{t\to\infty} \mu^*\big(C_\infty, C_\infty^t\big) = n - d_\infty, \tag{2.98}$$

where C_∞ and C_∞^t are the constant matrices in (2.75) *corresponding to Y and Y_t.*

In the last part of this subsection we compare the above results with the limiting cases of the results in Subsection 2.2.1 when we assume that system (H) is nonoscillatory and the right endpoint

$b \to \infty$. Equations (2.43)–(2.44) in Theorem 2.3 yield that for any conjoined bases Y and $\hat{Y}$ there exist the limits

$$\lim_{t\to\infty} \mu(Y(t), \hat{Y}(t)) = m_L(a,\infty) - \widehat{m}_L(a,\infty) + \mu(Y(a), \hat{Y}(a)), \tag{2.99}$$

$$\lim_{t\to\infty} \mu^*(Y(t), \hat{Y}(t)) = \mu^*(Y(a), \hat{Y}(a)) - m_R[a,\infty) + \widehat{m}_R[a,\infty), \tag{2.100}$$

Then the result in Theorem 2.27 shows, how these two limits can be evaluated explicitly without using the numbers of left proper focal points of Y and $\hat{Y}$ in (a,∞) or without using the numbers of right proper focal points of Y and $\hat{Y}$ in $[a,\infty)$. Moreover, the results in Theorem 2.20 yield interesting connections with the limits of the corresponding equalities in (2.49). First of all, it is not at all clear whether the limits

$$\lim_{b\to\infty} m_{Lb}(a,b], \quad \lim_{b\to\infty} m_{Rb}[a,b) \tag{2.101}$$

exist, and if they exist, then what are their values. Below we show that both of these limits indeed exist and that the first one is equal to $m_{L\infty}(a,\infty]$ as we would formally expect, but surprisingly the second one is not equal to $m_{R\infty}[a,\infty)$ in general. More precisely, we have

$$\lim_{b\to\infty} m_{Lb}(a,b] \overset{(2.49)}{=} \lim_{b\to\infty} m_{Ra}[a,b) = m_{Ra}[a,\infty) \overset{(2.84)}{=} m_{L\infty}(a,\infty],$$

$$\lim_{b\to\infty} m_{Rb}[a,b) \overset{(2.49)}{=} \lim_{b\to\infty} m_{La}(a,b] = m_{La}(a,\infty) \overset{(2.84)}{=} m_{R\infty}[a,\infty) - m_{La}(\infty).$$

The above calculation shows that the second limit in (2.101) is equal to the formally expected value $m_{R\infty}[a,\infty)$ only when $m_{La}(\infty) = 0$, i.e., only when the principal solution Y_a is antiprincipal at ∞ according to Definition 2.14 and the subsequent comments. Therefore, by taking the limit as $b \to \infty$ in the estimates in (2.50) for the left proper focal points we obtain the statement in (2.88). On the other hand, by taking the limit as $b \to \infty$ in the estimates in (2.51) for the right proper focal points we obtain the estimates

$$m_{R\infty}[a,\infty) - m_{La}(\infty) \le m_R[a,\infty) \le m_{Ra}[a,\infty). \tag{2.102}$$

In (2.102), the upper bound is optimal according to (2.87), while the lower bound is not in general optimal. More precisely, the lower bound in (2.102) is optimal if and only if $m_{La}(\infty) = 0$.

2.3.2. Singular Sturmian comparison theorems. In this is subsection we study a more general situation in the singular Sturmian theory on the unbounded interval $\mathcal{I} = [a,\infty)$. More precisely, we present the Sturmian comparison theorems for conjoined bases of two nonoscillatory systems (H) and ($\hat{\mathrm{H}}$), which satisfy the Sturmian majorant condition (2.17). These results were developed in [87, Sections 4–6], see Appendix E. We note that the results in this subsection hold also with the left endpoint $a = -\infty$ for the interval $\mathcal{I} = (-\infty, b]$, if we consider the corresponding concept of the minimal principal solution of (H) at minus infinity. A detailed analysis of this case is presented in [87, Section 6], see Appendix E. The situation for the Sturmian comparison theorems on the unbounded interval $\mathcal{I} = (-\infty,\infty)$ is slightly different and we comment on this case at the end of this subsection.

First we recall a comparison result for nonoscillatory systems (H) and ($\hat{\mathrm{H}}$) under the majorant condition (2.17), as well as the invariance of the nonoscillation for system (H) and the transformed system ($\tilde{\mathrm{H}}$). The latter result is based on the generalized reciprocity principle in [35, Theorem 2.2].

Proposition 2.29. *Assume that (2.17) and (2.18) hold on $\mathcal{I} = [a,\infty)$ and that system (H) is nonoscillatory. Then*

(i) *system ($\hat{\mathrm{H}}$) is nonoscillatory, and*
(ii) *for every symplectic fundamental matrix $\hat{Z}$ of ($\hat{\mathrm{H}}$) the transformed system ($\tilde{\mathrm{H}}$) with the coefficient matrix $\tilde{\mathcal{H}}(t)$ given in (2.70) is nonoscillatory.*

We denote by Y_∞, $\hat{Y}_\infty$, $\tilde{Y}_\infty$ the minimal principal solutions of nonoscillatory systems (H), ($\hat{\mathrm{H}}$), ($\tilde{\mathrm{H}}$) at infinity, respectively. Moreover, let Z_∞, $\hat{Z}_\infty$, $\tilde{Z}_\infty$ be the associated symplectic fundamental matrices defined in (2.74). In the remaining part of this subsection we will consider the transformed system

($\tilde{\text{H}}$) with respect to the transformation matrix $\hat{Z}_\infty$. The following result shows that under natural assumptions the minimal principal solution Y_∞ of (H) at infinity is transformed into the minimal principal solution $\tilde{Y}_\infty$ of ($\tilde{\text{H}}$) at infinity and vice versa.

Theorem 2.30. *Assume that (2.17) and (2.18) hold on $\mathcal{I} = [a, \infty)$ and that system (H) is nonoscillatory. Then the conjoined basis $\hat{Z}_\infty^{-1} Y_\infty$ is the minimal principal solution of ($\tilde{\text{H}}$) at infinity and the conjoined basis $\hat{Z}_\infty \tilde{Y}_\infty$ is the minimal principal solution of (H) at infinity. That is, the equalities $\hat{Z}_\infty^{-1} Y_\infty = \tilde{Y}_\infty K$ and $\hat{Z}_\infty \tilde{Y}_\infty = Y_\infty K^{-1}$ hold for some constant invertible $n \times n$ matrix K.*

The next result provides the singular Sturmian comparison theorem for two systems (H) and ($\hat{\text{H}}$) on the unbounded interval $\mathcal{I} = [a, \infty)$ satisfying the majorant condition (2.17). In particular, it generalizes the comparison formulas in Theorems 2.12 and 2.13, as well as the separation formulas in Theorem 2.17 to this case.

Theorem 2.31 (Singular Sturmian comparison theorem)**.** *Assume that (2.17) and (2.18) hold on $\mathcal{I} = [a, \infty)$ and that system (H) is nonoscillatory. Then for every conjoined basis Y of (H) and for every conjoined basis $\hat{Y}$ of ($\hat{\text{H}}$) we have*

$$m_L(a,\infty] - \widehat{m}_L(a,\infty] = \widetilde{m}_L(a,\infty] + \mu(\hat{Y}(a), \hat{Y}_\infty(a)) - \mu(Y(a), \hat{Y}_\infty(a)), \tag{2.103}$$

$$m_R[a,\infty) - \widehat{m}_R[a,\infty) = \widetilde{m}_R[a,\infty) + \mu^*(Y(a), \hat{Y}_\infty(a)) - \mu^*(\hat{Y}(a), \hat{Y}_\infty(a)), \tag{2.104}$$

where $\widetilde{m}_L(a,\infty]$ and $\widetilde{m}_R[a,\infty)$ are the numbers of left and right proper focal points of the conjoined basis $\tilde{Y} := \hat{Z}_\infty^{-1} Y$ of ($\tilde{\text{H}}$) in the indicated intervals.

In the following we present the special case of Theorem 2.31 for $\hat{Y} := \hat{Y}_\infty$. It is a generalization of estimate (2.26) to the uncontrollable systems (H) and ($\hat{\text{H}}$). At the same time it is an extensions of (2.67) and (2.69) to the case of $b = \infty$ with the fundamental matrix $\hat{Z}(t) := \hat{Z}_\infty(t)$.

Theorem 2.32 (Singular Sturmian comparison theorem)**.** *Assume that (2.17) and (2.18) hold on $\mathcal{I} = [a, \infty)$ and that system (H) is nonoscillatory. Then for every conjoined basis Y of (H) we have*

$$m_L(a,\infty] = \widehat{m}_{L\infty}(a,\infty] + \widetilde{m}_L(a,\infty] - \mu(Y(a), \hat{Y}_\infty(a)), \tag{2.105}$$

$$m_R[a,\infty) = \widehat{m}_{R\infty}[a,\infty) + \widetilde{m}_R[a,\infty) + \mu^*(Y(a), \hat{Y}_\infty(a)), \tag{2.106}$$

where $\widetilde{m}_L(a,\infty]$ and $\widetilde{m}_R[a,\infty)$ are the numbers of left and right proper focal points of the conjoined basis $\tilde{Y} := \hat{Z}_\infty^{-1} Y$ of ($\tilde{\text{H}}$) in the indicated intervals.

The results in Theorem 2.31 (or Theorem 2.32) allow to derive various estimates for the numbers of left and right proper focal points of conjoined bases of (H) and ($\hat{\text{H}}$). In particular, in the next result we show the exact relationship between the numbers of proper focal points of the (minimal) principal solutions Y_∞, $\hat{Y}_\infty$, $\tilde{Y}_\infty$ and Y_a, $\hat{Y}_a$, $\tilde{Y}_a$ of systems (H), ($\hat{\text{H}}$), ($\tilde{\text{H}}$).

Theorem 2.33. *Assume that (2.17) and (2.18) hold on $\mathcal{I} = [a, \infty)$ and that system (H) is nonoscillatory. Then we have*

$$m_{L\infty}(a,\infty] = \widehat{m}_{L\infty}(a,\infty] + \widetilde{m}_{L\infty}(a,\infty] - \mu(Y_\infty(a), \hat{Y}_\infty(a)), \tag{2.107}$$

$$m_{R\infty}[a,\infty) = \widehat{m}_{R\infty}[a,\infty) + \widetilde{m}_{R\infty}[a,\infty) + \mu^*(Y_\infty(a), \hat{Y}_\infty(a)), \tag{2.108}$$

$$m_{La}(a,\infty] = \widehat{m}_{La}(a,\infty] + \widetilde{m}_{La}(a,\infty] + \mu^*(Y_\infty(a), \hat{Y}_\infty(a)), \tag{2.109}$$

$$m_{Ra}[a,\infty) = \widehat{m}_{Ra}[a,\infty) + \widetilde{m}_{Ra}[a,\infty) - \mu(Y_\infty(a), \hat{Y}_\infty(a)), \tag{2.110}$$

The following result confirms the intuitively expected fact that, given the same initial conditions, conjoined bases of the majorant system (H) have in general more focal points than conjoined bases of the minorant system ($\hat{\text{H}}$). It is a generalization of the first part of [58, Corollary 7.3.2, pg. 196].

Theorem 2.34 (Singular Sturmian comparison theorem)**.** *Assume that (2.17) and (2.18) hold on $\mathcal{I} = [a, \infty)$ and that system (H) is nonoscillatory. Let Y and $\hat{Y}$ be conjoined bases of (H) and ($\hat{\mathrm{H}}$), respectively, such that $Y(a) = \hat{Y}(a)\, K$ for some invertible matrix K. Then*

$$m_L(a, \infty] - \widehat{m}_L(a, \infty] = \widetilde{m}_L(a, \infty] \geq 0, \tag{2.111}$$

$$m_R[a, \infty) - \widehat{m}_R[a, \infty) = \widetilde{m}_R[a, \infty) \geq 0, \tag{2.112}$$

where $\widetilde{m}_L(a, \infty]$ and $\widetilde{m}_R[a, \infty)$ are the numbers of left and right focal points of the conjoined basis $\tilde{Y} := \hat{Z}_\infty^{-1} Y$ of ($\tilde{\mathrm{H}}$) in the indicated intervals.

Next we compare the numbers of focal points of the (minimal) principal solutions Y_∞, $\hat{Y}_\infty$, $\tilde{Y}_\infty$ and Y_a, $\hat{Y}_a$, $\tilde{Y}_a$. We also provide universal lower and upper bounds for the numbers of focal points of conjoined bases of (H) and ($\hat{\mathrm{H}}$).

Theorem 2.35. *Assume that (2.17) and (2.18) hold on $\mathcal{I} = [a, \infty)$ and that system (H) is nonoscillatory. Then for every conjoined basis Y of (H) the estimates in (2.86) and (2.87) hold, where the lower and upper bounds satisfy*

$$\widehat{m}_{La}(a, \infty] + \widetilde{m}_{La}(a, \infty] \leq m_{La}(a, \infty], \quad m_{L\infty}(a, \infty] \leq \widehat{m}_{L\infty}(a, \infty] + \widetilde{m}_{L\infty}(a, \infty], \tag{2.113}$$

$$\widehat{m}_{R\infty}[a, \infty) + \widetilde{m}_{R\infty}[a, \infty) \leq m_{R\infty}[a, \infty), \quad m_{Ra}[a, \infty) \leq \widehat{m}_{Ra}[a, \infty) + \widetilde{m}_{Ra}[a, \infty). \tag{2.114}$$

Moreover, for every conjoined basis $\hat{Y}$ of ($\hat{\mathrm{H}}$) we have the estimates

$$\widehat{m}_{La}(a, \infty] \leq \widehat{m}_L(a, \infty] \leq \widehat{m}_{L\infty}(a, \infty] \leq m_{L\infty}(a, \infty], \tag{2.115}$$

$$\widehat{m}_{R\infty}[a, \infty) \leq \widehat{m}_R[a, \infty) \leq \widehat{m}_{Ra}[a, \infty) \leq m_{Ra}[a, \infty). \tag{2.116}$$

If assumptions (2.17) and (2.18) hold on the interval $\mathcal{I} = (-\infty, \infty)$ and system (H) is nonoscillatory, then for every conjoined basis Y of (H) we have the estimates

$$m_L(-\infty, \infty] \leq m_{L\infty}(-\infty, \infty] \leq \widehat{m}_{L\infty}(-\infty, \infty] + \widetilde{m}_{L\infty}(-\infty, \infty], \tag{2.117}$$

$$m_R(-\infty, \infty) \geq m_{R\infty}(-\infty, \infty) \geq \widehat{m}_{R\infty}(-\infty, \infty) + \widetilde{m}_{R\infty}(-\infty, \infty). \tag{2.118}$$

The inequalities in (2.117) follow from the singular Sturmian separation theorem (i.e., from (2.86) in Theorem 2.21) for system (H) with $a \to -\infty$ and from (2.107) with $a \to -\infty$ by dropping the last term with the comparative index. The inequalities in (2.118) follow from the singular Sturmian separation theorem (i.e., from (2.87) in Theorem 2.21) for system (H) with $a \to -\infty$ and from (2.108) with $a \to -\infty$ by dropping the last term with the dual comparative index. The lower bounds in (2.118) for the numbers of right proper focal points of Y in the interval $(-\infty, \infty)$ improve the lower bound $\widehat{m}_\infty(-\infty, \infty)$ obtained in (2.27) by Došlý and Kratz.

The final result in this subsection describes properties of the comparative indices $\mu\big(Y(t), \hat{Y}_\infty(t)\big)$ and $\mu^*\big(Y(t), \hat{Y}_\infty(t)\big)$ when $t \to \infty$ for a conjoined bases Y of (H), see [87, Theorems 4.6].

Theorem 2.36. *Assume that (2.17) and (2.18) hold on $\mathcal{I} = [a, \infty)$ and that system (H) is nonoscillatory. Then for every conjoined basis Y of (H) the limits*

$$\lim_{t\to\infty} \mu(Y(t), \hat{Y}_\infty(t)) \quad \textit{and} \quad \lim_{t\to\infty} \mu^*(Y(t), \hat{Y}_\infty(t)) \tag{2.119}$$

exist and satisfy

$$\lim_{t\to\infty} \mu(Y(t), \hat{Y}_\infty(t)) = \widetilde{m}_L(\infty) - m_L(\infty) + \widehat{m}_{L\infty}(\infty), \tag{2.120}$$

$$\lim_{t\to\infty} \mu^*(Y(t), \hat{Y}_\infty(t)) = 0, \tag{2.121}$$

where $\widetilde{m}_L(\infty)$ is the multiplicity of a focal point at infinity of the conjoined basis $\tilde{Y} := \hat{Z}_\infty^{-1} Y$ of the transformed system ($\tilde{\mathrm{H}}$).

We remark that equation (2.120) is an extension of formula (2.66) in Theorem 2.12 with $\hat{Y} := \hat{Y}_{t_0}$ to the case $t_0 = \infty$. At the same time it is a generalization of formula (2.95) in Theorem 2.27 with $\tilde{Y} := Y_\infty$ to two systems (H) and ($\hat{\mathrm{H}}$) satisfying (2.17). Note that equation (2.121) represents an extension to $t_0 = \infty$ of the left continuity of the dual comparative index $\mu^*(Y(t), \hat{Y}_{t_0}(t))$ at the point t_0 in Theorem 2.13.

In the last part of this subsection we comment on the above results in connection with the results in Subsection 2.2.2, when we assume that system (H) is nonoscillatory and the right endpoint $b \to \infty$. Equations (2.67) and (2.67) in Theorems 2.12 and 2.13 yield that for any conjoined bases Y and $\hat{Y}$ of systems (H) and (Ĥ), respectively, there exist the limits

$$\lim_{t\to\infty} \mu(Y(t), \hat{Y}(t)) = m_L(a,\infty) - \widehat{m}_L(a,\infty) - \widetilde{m}_L(a,\infty) + \mu(Y(a), \hat{Y}(a)), \tag{2.122}$$

$$\lim_{t\to\infty} \mu^*(Y(t), \hat{Y}(t)) = \mu^*(Y(a), \hat{Y}(a)) - m_R[a,\infty) + \widehat{m}_R[a,\infty) + \widetilde{m}_R[a,\infty), \tag{2.123}$$

where $\widetilde{m}_L(a,\infty)$ and $\widetilde{m}_R[a,\infty)$ are the numbers of left and right proper focal points of the conjoined basis $\tilde{Y} := \hat{Z}_\infty^{-1} Y$ of system (H̃) in the indicated intervals. Note that these numbers are finite according to Proposition 2.29. The result in Theorem 2.36 then shows that in the special case of $\hat{Y} = \hat{Y}_\infty$ these two limits can be evaluated explicitly.

When dealing with the singular Sturmian separation theorems on the interval $\mathcal{I} = (-\infty,\infty)$ in Subsection 2.3.1, we added together the corresponding Sturmian separation theorems on $\mathcal{I} = [a,\infty)$ and on $\mathcal{I} = (-\infty, a]$ to get the results on the interval $\mathcal{I} = (-\infty,\infty)$. The situation for the Sturmian comparison theorems on $\mathcal{I} = (-\infty,\infty)$ is different, since the results on the interval $\mathcal{I} = [a,\infty)$ in this subsection together with the corresponding results on the interval $\mathcal{I} = (-\infty, a]$ in general do not combine to obtain the Sturmian comparison theorems on the entire interval $\mathcal{I} = (-\infty,\infty)$. The main reason can be seen in Theorem 2.31, which implies that we need to employ two different transformation matrices $\hat{Z}_{\pm\infty}(t)$ in the neighborhoods of $\pm\infty$. Hence, we obtain two different transformation systems (H̃) – one on the interval $\mathcal{I} = [a,\infty)$ with the transformation matrix $\hat{Z}_\infty(t)$ and one on the interval $\mathcal{I} = (-\infty, a]$ with the transformation matrix $\hat{Z}_{-\infty}(t)$. Nevertheless, the results presented in this subsection remain valid also on the interval $\mathcal{I} = (-\infty,\infty)$ under the additional assumption that the minimal principal solutions $\hat{Y}_\infty$ and $\hat{Y}_{-\infty}$ at $\pm\infty$ of the minorant system (Ĥ) coincide, meaning that $\hat{Y}_{-\infty}$ is a constant nonsingular multiple of $\hat{Y}_\infty$ (which yields that $\hat{Z}_\infty(t) = \hat{Z}_{-\infty}(t)$).

2.3.3. Sturmian comparison theorems for controllable systems. In this subsection we discuss singular Sturmian comparison theorems for two linear Hamiltonian systems (H) and (Ĥ), satisfying the standard majorant condition (2.17) and the Legendre condition (2.18), in the the special setting. Namely, we consider the situation when

- the interval $\mathcal{I} = (a,\infty)$ is open, where the point a can be singular (including the case of $a = -\infty$),
- both systems (H) and (Ĥ) are completely controllable on the interval (a,∞) and nonoscillatory (at a and at infinity),
- the principal solution of the minorant system (Ĥ) at a is at the same time principal at infinity, i.e., $\hat{Y}_a = \hat{Y}_\infty$ holds.

The motivation of this kind of results comes from a singular Sturmian comparison theorem for the second order Sturm–Liouville differential equations, which was proven in [1,2] by Aharonov and Elias. More precisely, the result in [1, Theorem 1] or [2, Theorem 1] states that under a certain strict majorant condition on the coefficients every nontrivial solution of the majorant equation has a zero in the open interval (a,∞), if the principal solution at a of the minorant equation is principal also at infinity. This corresponds, in the spirit of Remark 2.16, to the situation when the principal solution of the minorant equation has zeros at a and at infinity, and hence the results in [1, Theorem 1] or [2, Theorem 1] can be interpreted from this point of view as the standard Sturmian comparison theorems on a compact interval $[a,b]$.

Comparing with Subsection 2.3.2 now we will consider the endpoint $a \in \mathbb{R} \cup \{-\infty\}$ to be *singular*, especially we allow $a = -\infty$. However, the results of this subsection are formulated in a way which includes both regular and singular endpoint a. For this purpose we will utilize the definition of the principal solution Y_a of (H) at a as a conjoined basis of (H) with the matrix $X_a(t)$ invertible on $(a,\beta]$

for some $\beta \in (a,\infty)$ and satisfying

$$\lim_{t\to a^+} \left(-\int_t^\beta X_a^{-1}(s)\, B(s)\, X_a^{T-1}(s)\,\mathrm{d}s \right)^{-1} = 0. \tag{2.124}$$

This unified definition (regarding with the definition of the principal solution at $\pm\infty$) is justified by the comments in [3, pg. 173] and by the results of [85, Theorem 5.8] and in Subsections 2.3.1 and 2.3.2. More precisely, the principal solution of (H) at a defined according to (2.124) coincides with the principal solution of (H) at the point a defined in (1.12), i.e., by the initial condition $Y_a(a) = E$, when the endpoint a is regular.

Following Theorem 2.15, for a conjoined basis Y of a nonoscillatory and completely controllable system (H) the multiplicity of its focal point at a and at ∞ is defined by the quantity

$$m(t_0) := \operatorname{def} W(Y_{t_0}, Y) = n - \operatorname{rank} W(Y_{t_0}, Y), \quad t_0 \in \{a, \infty\}, \tag{2.125}$$

where Y_∞ and Y_a are the principal solutions of system (H) at ∞ and at a, respectively, and where $W(Y_{t_0}, Y) = Y_{t_0}^T(t)\mathcal{J}Y(t)$ is the constant Wronskian of Y_{t_0} and Y. Definition (2.125) complies with the definition in (2.1), since for the point $t_0 \in (a,\infty)$ we have $\operatorname{rank} X(t_0) = \operatorname{rank} W(Y_{t_0}, Y)$, where Y_{t_0} is the principal solution of (H) at the point $t_0 \in (a,\infty)$.

The results below were published in [88, Section 3]. First we present formulas for the numbers of focal points of conjoined bases of systems (H) and ($\hat{\mathrm{H}}$) under the assumption $\hat{Y}_a = \hat{Y}_\infty$ for the principal solutions of ($\hat{\mathrm{H}}$) at a and ∞. This statement follows from Theorems 2.32 and 2.33.

Theorem 2.37 (Singular Sturmian comparison theorem)**.** *Assume that (2.17) and (2.18) hold on (a,∞) and that systems (H) and ($\hat{\mathrm{H}}$) are nonoscillatory and completely controllable. In addition, assume that $\hat{Y}_a = \hat{Y}_\infty$ holds. Then*

$$m_a(a,\infty] = \widehat{m}_a(a,\infty] + \widetilde{m}_{La}(a,\infty], \quad m_a[a,\infty) = \widehat{m}_a[a,\infty) + \widetilde{m}_{Ra}[a,\infty), \tag{2.126}$$

and for any conjoined basis Y of (H) we have the equality

$$m(a,\infty) = \operatorname{rank} W(Y_a, Y) + \widehat{m}_a(a,\infty) + \widetilde{m}_R[a,\infty), \tag{2.127}$$

where the quantity $\widetilde{m}_R[a,\infty)$ refers to the conjoined basis $\tilde{Y} := \hat{Z}_\infty^{-1} Y$ of ($\tilde{\mathrm{H}}$). In particular,

$$m(a,\infty) \geq m_a(a,\infty) = \widehat{m}_a(a,\infty) + \widetilde{m}_{Ra}[a,\infty). \tag{2.128}$$

As a consequence of Theorem 2.37 we obtain that conjoined bases of the majorant system (H), except of the principal solution Y_a of (H) at a (up to a right constant nonsingular multiple), have at least one focal point in the open interval (a,∞). This represents a direct generalization of the Sturmian comparison theorems in [1, Theorem 1] and [2, Theorem 1] by Aharonov and Elias.

Corollary 2.38. *Assume that (2.17) and (2.18) hold on (a,∞) and that systems (H) and ($\hat{\mathrm{H}}$) are nonoscillatory and completely controllable. If $\hat{Y}_a = \hat{Y}_\infty$ and Y is a conjoined basis of (H) such that $Y \neq Y_a$, then $m(a,\infty) \geq 1$.*

Remark 2.39. The equality in (2.128) yields the traditional comparison theorem for principal solutions Y_a and $\hat{Y}_a$. Namely, if Y_a has no focal points in (a,∞), then also $\hat{Y}_a$ has no focal points in (a,∞). Moreover, compared to inequality (2.26) the estimate in (2.128) provides an improved lower bound for the number of focal points in (a,∞) of any conjoined basis Y of (H) for the case when $\hat{Y}_a = \hat{Y}_\infty$.

In the following result we show that the number of focal points in (a,∞) of any conjoined basis Y of (H) is given by the rank of the Wronskian of Y with Y_a, when the principal solution Y_a of system (H) has no focal points in (a,∞).

Theorem 2.40. *Assume that (2.17) and (2.18) hold on (a,∞) and that systems (H) and ($\hat{\mathrm{H}}$) are nonoscillatory and completely controllable. In addition, assume that $\hat{Y}_a = \hat{Y}_\infty$ and that Y_a has no focal points in (a,∞). Then $Y_a = Y_\infty$ and for every conjoined basis Y of (H) we have the equalities*

$$m(a,\infty) = \operatorname{rank} W(Y_a, Y), \quad m(a) = \operatorname{def} W(Y_a, Y) = m(\infty). \tag{2.129}$$

Under the assumptions of Theorem 2.40, the equalities in (2.129) can be combined to obtain $m[a,\infty) = n = m(a,\infty]$ for every conjoined basis Y of (H). We note that the same conclusion $\widehat{m}[a,\infty) = n = \widehat{m}(a,\infty]$ for every conjoined basis $\hat{Y}$ of ($\hat{\text{H}}$) follows from the singular Sturmian separation theorem (Theorem 2.21) applied to the minorant system ($\hat{\text{H}}$) under the additional assumption $\hat{Y}_a = \hat{Y}_\infty$. Also, given that $\hat{Y}_a = \hat{Y}_\infty$ and $Y_a = Y_\infty$, it follows by the transformation result in Theorem 2.30 that the statement of Theorem 2.40 can be supplemented by the additional conclusion $\tilde{Y}_a = \tilde{Y}_\infty$. Note that when the point a is regular, then in (2.129) we have $m(a,\infty) = \operatorname{rank} X(a)$ and $m(a) = \operatorname{def} X(a)$, since in this case $W(Y_a, Y) = -X(a)$.

In the second main result of this subsection we complement the result in Theorem 2.37 (resp. in Remark 2.39) by providing an exact relationship between the considered properties of the principal solutions Y_a and $\hat{Y}_a$. This relationship is expressed in terms of the Riccati quotient associated with the principal solution $\hat{Y}_a$ of ($\hat{\text{H}}$).

Theorem 2.41 (Singular Sturmian comparison theorem)**.** *Assume that (2.17) and (2.18) hold on (a,∞) and that systems (H) and ($\hat{\text{H}}$) are nonoscillatory and completely controllable. In addition, assume that $\hat{Y}_a = \hat{Y}_\infty$ holds. Then the following statements are equivalent.*

(i) *The principal solution Y_a of (H) has no focal points in (a,∞).*

(ii) *The principal solution $\hat{Y}_a$ of ($\hat{\text{H}}$) has no focal points in (a,∞) and*

$$\mathcal{H}(t) - \hat{\mathcal{H}}(t) = \big(-\hat{Q}_a(t),\ I\big)^T [B(t) - \hat{B}(t)] \big(-\hat{Q}_a(t),\ I\big), \quad t \in (a,\infty), \tag{2.130}$$

where $\hat{Q}_a(t) := \hat{U}_a(t)\, \hat{X}_a^{-1}(t)$ is the symmetric Riccati quotient which corresponds to the principal solution $\hat{Y}_a$ of ($\hat{\text{H}}$) on (a,∞).

In this case, the symmetric Riccati quotient $Q_a(t) := U_a(t)\, X_a^{-1}(t)$ corresponding to the principal solution Y_a of (H) satisfies $Q_a(t) = \hat{Q}_a(t)$ on (a,∞).

The above results in Theorems 2.37, 2.40, and 2.41 are new even for the second order Sturm–Liouville differential equations. And even in this special case they generalize the Sturmian comparison theorems in [1, Theorem 1] and [2, Theorem 1] by Aharonov and Elias. More details on this subject are presented in [88, Section 4].

2.4. Further research directions

The research in the theory of Riccati matrix differential equations can be pursued in several directions. One possibility is to consider the discrete analogy, i.e., the Riccati matrix difference equations. These are important objects e.g. for discrete variational analysis or discrete filtering theory [5–7, 18], as well as for the discrete oscillation theory [15, 16, 27, 31, 97] or for numerical algorithms for computing the eigenvalues of symmetric banded matrices [61]. An interesting order preserving property of the discrete Riccati matrix equation was recently derived in [95] by Štoudková Růžičková, which is an analogue of the known order preserving property of the Riccati matrix differential equation (R), see the work by Stokes in [74, 75].

Open problems in the Sturmian theory of linear Hamiltonian systems include the development of this theory when we remove the Legendre condition (1.3), thus considering the oscillation numbers for conjoined bases of system (H) as it is presented in [39] by Elyseeva. This naturally leads to the connections between the Sturmian theory and the theory of Maslov index [10, 40, 51, 52]. In a close relationship with the latter research direction we mention the classical Gelfand–Lidskii–Yakubovich oscillation theory [44, 62, 98–100], to which we recently contributed by providing an explicit connection between the Lidskii angles of symplectic matrices with the comparative index [90]. Another open problem is to find explicit expressions for the limits of the comparative index in (2.122) and (2.123) for arbitrary conjoined bases Y of system (H) and $\hat{Y}$ of system ($\hat{\text{H}}$). Moreover, the question of the validity of the Sturmian comparison theorems for systems (H) and ($\hat{\text{H}}$) on the unbounded interval $\mathcal{I} = (-\infty,\infty)$ remains an open problem when the minimal principal solutions $\hat{Y}_\infty$ and $\hat{Y}_{-\infty}$ of ($\hat{\text{H}}$) at $\pm\infty$ differ, meaning that $\hat{Y}_{-\infty}$ is not a constant nonsingular multiple of $\hat{Y}_\infty$. In the spectral theory of linear Hamiltonian systems we may consider singular oscillation theorems on unbounded intervals [45] or applications of the theory of principal and antiprincipal solutions of (H) at infinity in the Weyl–Titchmarsh theory [94].

In general, we can say that it is indeed beneficial to develop the oscillation and spectral theory of linear Hamiltonian systems together with the oscillation and spectral theory of their discrete time counterparts, which are the symplectic difference systems – see the recent monograph [27] on this subject and the numerous references therein. This approach can be highlighted by developing the Sturmian theory for symplectic (or Hamiltonian) dynamic systems on time scales, such as in the references [4, 17, 28, 93, 101]. In particular, it is an open problem what is the multiplicity of a focal point of a conjoined basis in this abstract setting.

Declaration about co-authorship

Hereby I declare that the results in papers [84, 86, 87], which are contained in Appendices C–E of this habilitation thesis, were obtained in a joint co-authorship with prof. Roman Šimon Hilscher. The contribution of both authors in these publications was equal.

In Brno, May 21, 2021 Peter Šepitka

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APPENDIX A

Paper by Šepitka (JDDE 2020)

This paper entitled "Genera of conjoined bases for (non)oscillatory linear Hamiltonian systems: extended theory" appeared in the Journal of Dynamics and Differential Equations, 32 (2020), no. 3, 1139–1155, see item [78] in the bibliography.

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Genera of Conjoined Bases for (Non)oscillatory Linear Hamiltonian Systems: Extended Theory

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Abstract

In this paper we study the properties of conjoined bases of a general linear Hamiltonian system without any controllability condition. When the Legendre condition holds and the system is nonoscillatory, it is known from our previous work that conjoined bases with eventually the same image form a special structure called a genus. In this work we extend the theory of genera of conjoined bases to arbitrary systems, for which the Legendre condition is not assumed and/or the system may be oscillatory. We derive a classification of all genera of conjoined bases and show that they form a complete lattice. These results are based on the relationship between subspaces of solutions of a linear control system and orthogonal projectors satisfying a certain Riccati type differential equation. The presented theory is applied in our paper (Šepitka in Discrete Contin Dyn Syst 39(4):1685–1730, 2019) to general Riccati matrix differential equations for possibly uncontrollable linear Hamiltonian systems.

Keywords Linear Hamiltonian system · Genus of conjoined bases · Riccati differential equation · Controllability · Orthogonal projector

Mathematics Subject Classification 34C10

1 Introduction

Let $n \in \mathbb{N}$ be a given dimension and $A, B, C : [a, \infty) \to \mathbb{R}^{n\times n}$ be given piecewise continuous matrix-valued functions such that $B(t)$ and $C(t)$ are symmetric. In this paper we study the properties of conjoined bases of the linear Hamiltonian system

$$x' = A(t)\,x + B(t)\,u, \quad u' = C(t)\,x - A^T(t)\,u, \quad t \in [a, \infty). \tag{H}$$

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Classical theory of system (H) involves a complete controllability (or equivalently an identical normality) assumption, see e.g. [1,5,8,11,14,16]. Recently in [19–23] the author and Šimon Hilscher developed the theory of principal and antiprincipal solutions at infinity for a nonoscillatory and possibly abnormal system (H) satisfying the Legendre condition

$$B(t) \geq 0 \quad \text{for all } t \in [a, \infty). \tag{1.1}$$

We showed the existence of principal solutions $(\hat{X}, \hat{U})$ at infinity with all ranks of $\hat{X}(t)$ in the range between $n - d_\infty$ and n, where d_∞ is the maximal order of abnormality of (H) (defined in (2.6) below), and derived their classification and limit properties with antiprincipal solutions at infinity. These results motivated the investigations in [20,22], where we studied conjoined bases (X, U) of (H) with eventually the same image of $X(t)$. According to [20, Definition 6.3] we say that such conjoined bases of (H) form a *genus* $\mathcal{G}$. We proved that every genus $\mathcal{G}$ can be represented by an orthogonal projector $R_\mathcal{G}(t)$ satisfying the Riccati type matrix differential equation

$$R_\mathcal{G}' - A(t)\, R_\mathcal{G} - R_\mathcal{G}\, A^T(t) + R_\mathcal{G}\, [A(t) + A^T(t)]\, R_\mathcal{G} = 0, \quad t \in [a, \infty). \tag{1.2}$$

This allowed to obtain under (1.1) a geometric description of the set Γ of all genera of conjoined bases of a nonoscillatory system (H), being a complete lattice [22, Theorem 4.8].

In this paper we extend the theory of genera of conjoined bases to arbitrary systems (H), for which the Legendre condition is not assumed and/or the system may be oscillatory. More precisely, in the new general definition of a genus corresponding to a conjoined basis (X, U) of (H) we consider the subspace

$$\operatorname{Im} X(t) + \operatorname{Im} R_{\Lambda\infty}(t), \tag{1.3}$$

where $R_{\Lambda\infty}(t)$ is the orthogonal projector onto the maximal subspace of the values at t of eventually degenerate solutions $(x \equiv 0, u)$ of (H). We refer to Sect. 2 for precise definitions of these notions, including the notions of a conjoined basis and the (non)oscillation of system (H). When the Legendre condition (1.1) holds and system (H) is nonoscillatory, then the subspace in (1.3) eventually coincides with the image of $X(t)$, which yields the previous definition of a genus of conjoined bases in [20, Definition 6.3] for this special case.

In the more general context of (1.3) we then derive (Theorems 4.7 and 4.8) a characterization of a genus $\mathcal{G}$ of conjoined bases in terms of the orthogonal projector $R_\mathcal{G}(t)$ onto the subspace in (1.3), where $R_\mathcal{G}(t)$ satisfies the Riccati equation (1.2). This then leads to a natural ordering on the set Γ of all genera of conjoined bases (Theorem 4.12), as well as to a result stating that the set Γ forms a complete lattice (Theorem 4.14). This way we obtain direct generalizations of the results in [22] to the case of system (H) without the Legendre condition (1.1) or system (H) which is allowed to be oscillatory. The above results are based on the analysis (Theorems 3.9, 3.10, and 3.13) of the relationship between the orthogonal projectors solving the Riccati equation (1.2) and certain subspaces associated to the linear control system

$$x' = A(t)\, x + B(t)\, u, \quad t \in [a, \infty). \tag{1.4}$$

In this analysis we make use of the properties of filters in ordered sets (lattices).

The presented theory of genera of conjoined bases (in Sect. 4) is interesting by itself, as well as it is motivating for subsequent research. It is applied in our subsequent paper [18, Theorem 7.8 and Remark 7.9] in the study of distinguished solutions of Riccati matrix differential equations for possibly abnormal linear Hamiltonian systems (compare with the theory of Riccati matrix equations in [5,14,15]). Reference [18] appeared during the review process of this paper. Moreover, system (H) with the maximal order of abnormality $d_\infty = 0$

satisfies the condition of weak disconjugacy, see [7, Definition 2.2], [9, Remark 2.6, part 2], and [10, Lemma 5.5], so that the presented theory can be applied also to weakly disconjugate system (H).

The paper is organized as follows. In Sect. 2 we recall basic notions from the theory of linear Hamiltonian systems. In Sect. 3 we study the properties of solutions of the linear control system (1.4) with the aid of the Riccati equation (1.2). Finally, in Sect. 4 we present the main results of this paper regarding the new concept of a genus of conjoined bases of (H). We also provide several examples illustrating this new theory.

2 Linear Hamiltonian Systems and Their Solutions

In this section we review needed notions and results about linear Hamiltonian systems. Following a common convention, matrix solutions (X, U) of (H) will be denoted by the capital letters, where $X, U : [a, \infty) \to \mathbb{R}^{n\times n}$ are piecewise continuously differentiable matrix-valued functions on $[a, \infty)$. A solution (X, U) of (H) is called a *conjoined basis* if $\operatorname{rank}\,(X^T(t),\ U^T(t))^T = n$ and $X^T(t)\,U(t)$ is symmetric at some and hence at any $t \in [a, \infty)$. The principal solution $(\hat{X}_\alpha, \hat{U}_\alpha)$ at the point $\alpha \in [a, \infty)$ is an example of such a conjoined basis. It is defined as the solution of (H) with the initial conditions $\hat{X}_\alpha(\alpha) = 0$ and $\hat{U}_\alpha(\alpha) = I$.

The oscillation of conjoined bases of (H) satisfying (1.1) is defined via the concept of proper focal points, see [27, Definition 1.1] and [6,24,25]. However, this concept will not be explicitly needed in this paper. By [26, Definition 2.1], a conjoined basis (X, U) of (H) is called *nonoscillatory* if there exists $\alpha \in [a, \infty)$ such that $\operatorname{Ker} X(t)$ is constant on $[\alpha, \infty)$. In the opposite case (X, U) is called *oscillatory*. The main result of [26] then describes the nonoscillatory behavior of conjoined bases of (H).

Proposition 2.1 *Assume that the Legendre condition* (1.1) *holds. Then there exists a nonoscillatory conjoined basis of* (H) *if and only if every conjoined basis of* (H) *is nonoscillatory.*

Based on this result we say that system (H) is (*non*)*oscillatory* if one and hence all conjoined bases of (H) are (non)oscillatory.

In this paper we will use orthogonal projectors. If $\mathcal{V}$ is a subspace of $\mathbb{R}^n$, then we denote by $\mathcal{P}_\mathcal{V}$ the orthogonal projector onto $\mathcal{V}$. That is, $\mathcal{P}_\mathcal{V}$ is a symmetric and idempotent $n \times n$ matrix such that $\operatorname{Im} \mathcal{P}_\mathcal{V} = \mathcal{V} = \operatorname{Ker}\,(I - \mathcal{P}_\mathcal{V})$ and $\operatorname{Ker} \mathcal{P}_\mathcal{V} = \mathcal{V}^\perp = \operatorname{Im}\,(I - \mathcal{P}_\mathcal{V})$. Orthogonal projectors are easily constructed by using Moore–Penrose pseudoinverses. More precisely, for a matrix $M \in \mathbb{R}^{m\times n}$ we denote by $M^\dagger \in \mathbb{R}^{n\times m}$ its Moore–Penrose pseudoinverse. Then the matrix $MM^\dagger$ is the orthogonal projector onto $\operatorname{Im} M$ and the matrix $M^\dagger M$ is the orthogonal projector onto $\operatorname{Im} M^\dagger = \operatorname{Im} M^T$. Moreover, $\operatorname{rank} M = \operatorname{rank} MM^\dagger = \operatorname{rank} M^\dagger M$. For a general theory of pseudoinverse matrices we refer to [2], [3, Chapter 6], and [4, Section 1.4].

Given a conjoined basis (X, U) of (H), by its *kernel*, resp. *image*, we mean the kernel, resp. image, of X. Furthermore, we define on $[a, \infty)$ the orthogonal projectors onto the subspaces $\operatorname{Im} X^T(t)$ and $\operatorname{Im} X(t)$ by

$$P(t) := \mathcal{P}_{\operatorname{Im} X^T(t)} = X^\dagger(t)\,X(t), \quad R(t) := \mathcal{P}_{\operatorname{Im} X(t)} = X(t)\,X^\dagger(t). \tag{2.1}$$

If (X, U) has constant kernel on $[\alpha, \infty) \subseteq [a, \infty)$, then $P(t)$ is constant on $[\alpha, \infty)$ and we set

$$P := P(t) \quad \text{on } [\alpha, \infty). \tag{2.2}$$

In this case (X, U) has constant rank r on $[\alpha, \infty)$ with

$$r := \operatorname{rank} X(t) = \operatorname{rank} P = \operatorname{rank} R(t) \quad \text{on } [\alpha, \infty). \tag{2.3}$$

The following result is from [19, Theorem 4.2], in which we observe that condition (1.1) is not needed.

Proposition 2.2 *Let (X, U) be a conjoined basis of (H) with constant kernel on $[\alpha, \infty)$ and let P and $R(t)$ be the corresponding matrices in (2.2) and (2.1). Then the equalities*

$$\operatorname{Im} [U(t)\,(I - P)] = \operatorname{Ker} R(t), \quad B(t) = R(t)\,B(t) = B(t)\,R(t) \tag{2.4}$$

hold for all $t \in [\alpha, \infty)$. Moreover, the matrix $R(t)$ solves the Riccati equation (1.2) on $[\alpha, \infty)$.

Following the standard notation used in [13, Section 3] and [22, Section 2], we denote by $\Lambda[\alpha, \infty)$ the linear space of n-dimensional piecewise continuously differentiable vector-valued functions u which satisfy the equations $u' = -A^T(t)\,u$ and $B(t)\,u = 0$ on $[\alpha, \infty)$. The functions $u \in \Lambda[\alpha, \infty)$ then correspond to the solutions $(x \equiv 0, u)$ of system (H) on $[\alpha, \infty)$. The space $\Lambda[\alpha, \infty)$ is finite-dimensional with $d[\alpha, \infty) := \dim \Lambda[\alpha, \infty) \leq n$. The number $d[\alpha, \infty)$ is called the *order of abnormality* of system (H) on the interval $[\alpha, \infty)$. For a given $\alpha \in [a, \infty)$ we denote by $\Lambda_t[\alpha, \infty)$ the subspace in $\mathbb{R}^n$ of values of functions $u \in \Lambda[\alpha, \infty)$ at the point $t \in [\alpha, \infty)$, i.e.,

$$\Lambda_t[\alpha, \infty) := \{c \in \mathbb{R}^n, \quad u(t) = c \quad \text{for some } u \in \Lambda[\alpha, \infty)\}, \quad t \in [\alpha, \infty). \tag{2.5}$$

It is easy to see that the subspace $\Lambda_t[\alpha, \infty)$ is finite-dimensional with $\dim \Lambda_t[\alpha, \infty) = d[\alpha, \infty)$ for all $t \in [\alpha, \infty)$. We note that the set $\Lambda[t, \infty)$ is nondecreasing in t on $[a, \infty)$ with respect to the usual ordering in the set of linear subspaces of $\mathbb{R}^n$ and hence it is eventually constant. This means that the integer-valued function $d[t, \infty)$ is nondecreasing, piecewise constant, and right-continuous on $[a, \infty)$. In particular, there exists the limit

$$d_\infty := \lim_{t \to \infty} d[t, \infty) = \max_{t \in [a, \infty)} d[t, \infty), \quad 0 \leq d_\infty \leq n, \tag{2.6}$$

which we call the *maximal order of abnormality* of (H). Moreover, the definition of the quantity d_∞ in (2.6) and the monotonicity of the function $d[t, \infty)$ then imply the existence of the point $\alpha_\infty \in [a, \infty)$ satisfying

$$\alpha_\infty = \min\{\alpha \in [a, \infty), \quad d[\alpha, \infty) = d_\infty\}. \tag{2.7}$$

From (2.6) and (2.7) we then obtain that the subspace $\Lambda[\alpha_\infty, \infty)$ satisfies the equalities

$$\Lambda[\alpha_\infty, \infty) = \lim_{\alpha \to \infty} \Lambda[\alpha, \infty) = \max_{\alpha \in [a, \infty)} \Lambda[\alpha, \infty), \tag{2.8}$$

$$\Lambda[\alpha, \infty) \equiv \Lambda[\alpha_\infty, \infty), \quad \alpha \in [\alpha_\infty, \infty). \tag{2.9}$$

3 Auxiliary Results on Linear Control Systems

In this section we present some auxiliary results about the solutions of the first order linear differential equation (1.4), where $A, B : [a, \infty) \to \mathbb{R}^{n \times n}$ are given piecewise continuous matrix-valued functions. Our main results describe the relationship between the subspaces of values of the solutions of system (1.4) and the orthogonal projectors solving the Riccati equation (1.2), see Theorems 3.9 and 3.13. These properties will be utilized in Sect. 4 in order to develop correctly the concept of a genus of conjoined bases of system (H).

Solutions of (1.4) are considered to be pairs of n-dimensional vector-valued functions (x, u) such that $u(t)$ is piecewise continuous on $[a, \infty)$ and $x(t)$ is piecewise continuously differentiable on $[a, \infty)$. In the literature such a pair (x, u) is sometimes termed as *admissible*, while (1.4) is then called the equation of motion. This terminology is motivated by the variational analysis and control theory, where the solutions of (1.4) are related with a certain type of quadratic functionals, see e.g. [5,11,12,14,17]. By the symbol $\mathcal{S}$ we will denote the linear space of the first components of solutions of (1.4), i.e.,

$$\mathcal{S} := \{x : [a, \infty) \to \mathbb{R}^n,\ (x, u) \text{ is admissible for some } u\}.$$

Let $t \in [a, \infty)$. For a given subspace $\mathcal{V} \subseteq \mathcal{S}$ we will denote by $\mathcal{V}_t$ the subspace of $\mathbb{R}^n$ consisting of the values of functions $x \in \mathcal{V}$ at the point t, that is

$$\mathcal{V}_t := \{c \in \mathbb{R}^n, \quad x(t) = c \quad \text{for some } x \in \mathcal{V}\}. \tag{3.1}$$

The main results of this section are formulated in terms of orthogonal projectors $Z(t)$ which solve the symmetric Riccati matrix differential equation (1.2), i.e.,

$$Z' - A(t)\, Z - Z\, A^T(t) + Z\, [A(t) + A^T(t)]\, Z = 0, \quad t \in [a, \infty). \tag{3.2}$$

We recall from Sect. 2 that the matrix $M \in \mathbb{R}^{n\times n}$ is an orthogonal projector if M is symmetric and idempotent, i.e., the equalities $M = M^T$ and $M^2 = M$ hold. In the following three propositions and one remark we collect basic results about the solutions $Z(t)$ of (3.2) which are orthogonal projectors for all $t \in [a, \infty)$, see [22, Section 3].

Proposition 3.1 *Let $Z_0 \in \mathbb{R}^{n\times n}$ be an orthogonal projector and let $t_0 \in [a, \infty)$ be fixed. Then the unique solution $Z(t)$ of (3.2) which satisfies the initial condition $Z(t_0) = Z_0$ exists on the whole interval $[a, \infty)$ and the matrix $Z(t)$ is an orthogonal projector for all $t \in [a, \infty)$.*

Proposition 3.2 *Let $Z_1(t)$ and $Z_2(t)$ be two orthogonal projectors which solve (3.2) on $[a, \infty)$. Then the inclusion $\operatorname{Im} Z_1(t) \subseteq \operatorname{Im} Z_2(t)$ holds for every $t \in [a, \infty)$ if and only if the inclusion $\operatorname{Im} Z_1(t_0) \subseteq \operatorname{Im} Z_2(t_0)$ holds for some $t_0 \in [a, \infty)$.*

Proposition 3.3 *Let $Z_1(t)$ and $Z_2(t)$ be two orthogonal projectors which solve (3.2) on $[a, \infty)$. For each $t \in [a, \infty)$ denote by $\hat{Z}(t)$ and $\tilde{Z}(t)$ the orthogonal projectors onto the subspaces $\operatorname{Im} Z_1(t) \cap \operatorname{Im} Z_2(t)$ and $\operatorname{Im} Z_1(t) + \operatorname{Im} Z_2(t)$, respectively. Then the matrices $\hat{Z}(t)$ and $\tilde{Z}(t)$ satisfy (3.2) on $[a, \infty)$.*

Remark 3.4 The matrix $Z(t)$ is a solution of (3.2) if and only if $I - Z(t)$ is a solution of the Riccati equation

$$Y' + A^T(t)\, Y + Y\, A(t) - Y\, [A(t) + A^T(t)]\, Y = 0, \quad t \in [a, \infty), \tag{3.3}$$

which is obtained from (3.2) by changing the coefficient $A(t)$ to $-A^T(t)$.

Throughout this section we will use the following notation. By the symbol $\mathcal{L}(A)$ we will denote the set of all orthogonal projectors $Z(t)$ which solve (3.2) on the whole interval $[a, \infty)$. Furthermore, the symbol $\mathcal{L}(A, B)$ will denote the set of all orthogonal projectors $Z(t)$ which solve (3.2) on $[a, \infty)$ and in addition satisfy the inclusion

$$\operatorname{Im} B(t) \subseteq \operatorname{Im} Z(t) \quad \text{for all } t \in [a, \infty). \tag{3.4}$$

Remark 3.5 (i) The results in Propositions 3.1–3.3 show that the set $\mathcal{L}(A)$ can be ordered with respect to inclusion of subspaces $\operatorname{Im} Z(t)$. More precisely, $\mathcal{L}(A)$ is a complete lattice with the least element $Z(t) \equiv 0$ on $[a, \infty)$ and the greatest element $Z(t) \equiv I$ on $[a, \infty)$. In particular, the matrices $\hat{Z}(t)$ and $\tilde{Z}(t)$ from Proposition 3.3 represent the infimum and supremum of $Z_1(t)$ and $Z_2(t)$, respectively, in this ordering. We also note that the lattice $\mathcal{L}(A)$ is isomorphic to the complete lattice of all subspaces in $\mathbb{R}^n$.

(ii) The observation in Remark 3.4 implies that the complete lattice $\mathcal{L}(-A^T)$ associated to equation (3.3) is isomorphic to the dual lattice to $\mathcal{L}(A)$. In particular, the orthogonal projector $Z(t)$ belongs to $\mathcal{L}(A)$ if and only if the orthogonal projector $I - Z(t)$ belongs to $\mathcal{L}(-A^T)$.

From Remark 3.5(i) and the obvious relation $\mathcal{L}(A, B) \subseteq \mathcal{L}(A)$ we obtain immediately that the matrices $Z(t)$ belonging to set $\mathcal{L}(A, B)$ can be also ordered with respect to inclusion of the subspaces $\operatorname{Im} Z(t)$. Moreover, in the next proposition we prove that the ordered set $\mathcal{L}(A, B)$ is a sublattice of $\mathcal{L}(A)$.

Proposition 3.6 *The ordered set $\mathcal{L}(A, B)$ is a sublattice of the complete lattice $\mathcal{L}(A)$.*

Proof Let $Z_1(t)$ and $Z_2(t)$ be two orthogonal projectors belonging to $\mathcal{L}(A, B)$. Moreover, let $\hat{Z}(t)$ and $\tilde{Z}(t)$ be the orthogonal projectors in Proposition 3.3 and Remark 3.5(i) which respectively correspond to the infimum and supremum of $Z_1(t)$ and $Z_2(t)$ in the lattice $\mathcal{L}(A)$. We will show that the matrices $\tilde{Z}(t)$ and $\hat{Z}(t)$ belong to the set $\mathcal{L}(A, B)$. Indeed, the two projectors $\hat{Z}(t)$ and $\tilde{Z}(t)$ solve (3.2) on the whole interval $[a, \infty)$, by Proposition 3.3. Moreover, the inclusions $\operatorname{Im} B(t) \subseteq \operatorname{Im} Z_1(t)$ and $\operatorname{Im} B(t) \subseteq \operatorname{Im} Z_2(t)$ on $[a, \infty)$ imply that

$$\operatorname{Im} B(t) \subseteq \operatorname{Im} Z_1(t) \cap \operatorname{Im} Z_2(t) = \operatorname{Im} \hat{Z}(t), \quad \operatorname{Im} B(t) \subseteq \operatorname{Im} Z_1(t) + \operatorname{Im} Z_2(t) = \operatorname{Im} \tilde{Z}(t)$$

for every $t \in [a, \infty)$. This shows that the matrices $\hat{Z}(t)$ and $\tilde{Z}(t)$ satisfy condition (3.4) and hence, they belong to $\mathcal{L}(A, B)$. Therefore, the ordered set $\mathcal{L}(A, B)$ is a sublattice of the lattice $\mathcal{L}(A)$ and the proof is complete. □

In the following lemma we derive an auxiliary property of solutions of (1.4).

Lemma 3.7 *Let (x, u) be an admissible pair and let $Z(t)$ be an orthogonal projector belonging to $\mathcal{L}(A)$. Then the vector $w(t) := [I - Z(t)]\, x(t)$ satisfies on $[a, \infty)$ the equation*

$$w' - \left\{A(t) - Z(t)\, [A(t) + A^T(t)]\right\} w - [I - Z(t)]\, B(t)\, u(t) = 0. \tag{3.5}$$

Proof Let (x, u) and $Z(t)$ be as in the statement. By using (1.4) and (3.2) we have on $[a, \infty)$

$$\begin{aligned} w' &= -Z'x + (I - Z)\, x' = [-AZ - ZA^T + Z\,(A + A^T)\, Z]\, x + (I - Z)\,(Ax + Bu) \\ &= \left\{A - Z\,[A + A^T]\right\} w + (I - Z)\, Bu. \end{aligned}$$

Thus, the vector $w(t)$ satisfies the linear differential equation (3.5) on $[a, \infty)$. □

Remark 3.8 Let $t_0 \in [a, \infty)$ be fixed. Using the variation of constant formula applied to (3.5) we obtain for the function $w(t)$ in Lemma 3.7 the expression

$$w(t) = \Phi(t, t_0)\, w(t_0) + \Phi(t, t_0) \int_{t_0}^{t} \Phi^{-1}(s, t_0)\, [I - Z(s)]\, B(s)\, u(s)\, \mathrm{d}s, \quad t \in [a, \infty), \tag{3.6}$$

where $\Phi(t, t_0)$ is the fundamental matrix of the equation $y' = \{A(t) - Z(t)\, [A(t) + A^T(t)]\}\, y$ for $t \in [a, \infty)$ with $\Phi(t_0, t_0) = I$.

In the first main result of this section we provide a fundamental connection between the structure of the lattice $\mathcal{L}(A, B)$ and the subspaces $\mathcal{V}_t$ in (3.1). For this purpose we now recall the standard terminology for lattices $(\mathcal{M}, \preceq)$. Namely, a subset $\mathcal{F} \subseteq \mathcal{M}$ is called a *filter* of the lattice $\mathcal{M}$ if $\mathcal{F}$ is a sublattice of $\mathcal{M}$ such that for any $f \in \mathcal{F}$ and $x \in \mathcal{M}$ the condition $f \preceq x$ implies $x \in \mathcal{F}$. Moreover, for a given $f \in \mathcal{M}$ the subset $\mathcal{F} := \{x \in \mathcal{M},\ f \preceq x\}$ is called the *principal filter* of $\mathcal{M}$ *generated by* f. In particular, we describe all principal filters of $\mathcal{L}(A, B)$ in terms of the subspaces $\mathcal{V}_t$.

Theorem 3.9 *Let $Z(t)$ be an orthogonal projector belonging to the lattice $\mathcal{L}(A, B)$ and let $\mathcal{Z}$ be the principal filter of $\mathcal{L}(A, B)$ generated by $Z(t)$. Then the following statements hold.*

- (i) *For any given subspace $\mathcal{V} \subseteq \mathcal{S}$ the orthogonal projector $\tilde{Z}(t)$ onto the set $\mathcal{V}_t + \operatorname{Im} Z(t)$ belongs to $\mathcal{Z}$.*
- (ii) *For any given orthogonal projector $\tilde{Z}(t)$ from $\mathcal{Z}$ there exists a subspace $\mathcal{V} \subseteq \mathcal{S}$ such that the matrix $\tilde{Z}(t)$ is the orthogonal projector onto the set $\mathcal{V}_t + \operatorname{Im} Z(t)$ for all $t \in [a, \infty)$.*

Proof (i) Let $\mathcal{V}$ be a subspace of $\mathcal{S}$ and let $\tilde{Z}(t)$ be the orthogonal projector onto the subspace $\mathcal{V}_t + \operatorname{Im} Z(t)$ on $[a, \infty)$. Fix $t_0 \in [a, \infty)$ and consider the solution $\bar{Z}(t)$ of (3.2) satisfying the initial condition $\bar{Z}(t_0) = \tilde{Z}(t_0)$. From Proposition 3.1 it then follows that $\bar{Z}(t)$ is defined on the whole interval $[a, \infty)$ and the matrix $\bar{Z}(t)$ is an orthogonal projector for every $t \in [a, \infty)$. Moreover, the equalities $\mathcal{V}_{t_0} + \operatorname{Im} Z(t_0) = \operatorname{Im} \tilde{Z}(t_0) = \operatorname{Im} \bar{Z}(t_0)$ imply that $\mathcal{V}_{t_0} \subseteq \operatorname{Im} \bar{Z}(t_0)$ and $\operatorname{Im} Z(t_0) \subseteq \operatorname{Im} \bar{Z}(t_0)$. The latter inclusion then yields $\operatorname{Im} Z(t) \subseteq \operatorname{Im} \bar{Z}(t)$ for all $t \in [a, \infty)$, by Proposition 3.1. Let (x, u) be an admissible pair such that $x \in \mathcal{V}$. Since $x(t_0) \in \mathcal{V}_{t_0} \subseteq \operatorname{Im} \bar{Z}(t_0)$, the vector $w(t) := [I - \bar{Z}(t)]\, x(t)$ satisfies $w(t_0) = 0$. Consequently, using Lemma 3.7, formula (3.6), and condition (3.4) with $Z := \bar{Z}$ we have that $w(t) = 0$ for all $t \in [a, \infty)$. Therefore, the vector $x(t) \in \operatorname{Im} \bar{Z}(t)$ for every $t \in [a, \infty)$. This shows the inclusion $\mathcal{V}_t \subseteq \operatorname{Im} \bar{Z}(t)$ on $[a, \infty)$, which in turn implies that $\operatorname{Im} \tilde{Z}(t) = \mathcal{V}_t + \operatorname{Im} Z(t) \subseteq \operatorname{Im} \bar{Z}(t)$ for all $t \in [a, \infty)$. Suppose now that there exists $\tau \in [a, \infty)$ such that $\operatorname{Im} \tilde{Z}(\tau) \subsetneqq \operatorname{Im} \bar{Z}(\tau)$. Let $Z_\tau(t)$ be the solution of (3.2) satisfying the initial condition $Z_\tau(\tau) = \tilde{Z}(\tau)$. By using the similar arguments as above we obtain that $Z_\tau(t)$ can be extended to the whole interval $[a, \infty)$ being an orthogonal projector and the inclusion $\mathcal{V}_t + \operatorname{Im} Z(t) \subseteq \operatorname{Im} Z_\tau(t)$ holds for all $t \in [a, \infty)$. Moreover, the initial inclusion $\operatorname{Im} Z_\tau(\tau) \subsetneqq \operatorname{Im} \bar{Z}(\tau)$ yields the relation $\operatorname{Im} Z_\tau(t) \subsetneqq \operatorname{Im} \bar{Z}(t)$ on $[a, \infty)$ by Proposition 3.2. In particular, the choice $t = t_0$ implies that

$$\operatorname{Im} \bar{Z}(t_0) = \operatorname{Im} \tilde{Z}(t_0) = \mathcal{V}_{t_0} + \operatorname{Im} Z(t_0) \subseteq \operatorname{Im} Z_\tau(t_0) \subsetneqq \operatorname{Im} \bar{Z}(t_0),$$

which is a contradiction. Therefore, we have the equality $\operatorname{Im} \tilde{Z}(t) = \operatorname{Im} \bar{Z}(t)$ for all $t \in [a, \infty)$, which by the uniqueness of orthogonal projectors means that $\tilde{Z}(t) = \bar{Z}(t)$ on $[a, \infty)$. Hence, the matrix $\tilde{Z}(t)$ solves (3.2). Finally, the inclusions $\operatorname{Im} B(t) \subseteq \operatorname{Im} Z(t) \subseteq \operatorname{Im} \tilde{Z}(t)$ on $[a, \infty)$ yield that the orthogonal projector $\tilde{Z}(t)$ satisfies condition (3.4) and belongs to $\mathcal{Z}$.

(ii) Let $\tilde{Z}(t)$ be an orthogonal projector of the principal filter $\mathcal{Z}$. This means that the matrix $\tilde{Z}(t)$ solves (3.2) and that the inclusions $\operatorname{Im} B(t) \subseteq \operatorname{Im} Z(t) \subseteq \operatorname{Im} \tilde{Z}(t)$ hold on $[a, \infty)$. For a fixed $t_0 \in [a, \infty)$ consider a subspace $\mathcal{V} \subseteq \mathcal{S}$ satisfying the initial condition $\mathcal{V}_{t_0} = \operatorname{Im} \tilde{Z}(t_0)$. Using similar arguments to those of the proof of part (i) above we obtain that $\mathcal{V}_t \subseteq \operatorname{Im} \tilde{Z}(t)$ for all $t \in [a, \infty)$. Denote by $\bar{Z}(t)$ the orthogonal projector onto the subspace $\mathcal{V}_t + \operatorname{Im} Z(t)$ for each $t \in [a, \infty)$. Then the inclusion $\operatorname{Im} \bar{Z}(t) = \mathcal{V}_t + \operatorname{Im} Z(t) \subseteq \operatorname{Im} \tilde{Z}(t)$ holds on $[a, \infty)$. In particular, at the point $t = t_0$ we have that $\operatorname{Im} \tilde{Z}(t_0) = \mathcal{V}_{t_0} \subseteq \operatorname{Im} \bar{Z}(t_0)$. Hence, we get the equality $\operatorname{Im} \bar{Z}(t_0) = \operatorname{Im} \tilde{Z}(t_0)$, which by the uniqueness of orthogonal projectors means that $\bar{Z}(t_0) = \tilde{Z}(t_0)$. On the other hand, from part (i) of the theorem it follows that the matrix $\bar{Z}(t)$ solves (3.2) on $[a, \infty)$. Therefore, by Proposition 3.1 we conclude that $\bar{Z}(t) = \tilde{Z}(t)$

for all $t \in [a, \infty)$. Finally, the matrix $\tilde{Z}(t)$ is the orthogonal projector onto the subspace $\mathcal{V}_t + \operatorname{Im} Z(t)$ for every $t \in [a, \infty)$ and the proof is complete. □

From Proposition 3.6 we know that the set $\mathcal{L}(A, B)$, as a sublattice of the complete lattice $\mathcal{L}(A)$, is a complete lattice itself. The statements of Theorem 3.9 are then particularly important when the orthogonal projector $Z(t)$ is equal to the least element of $\mathcal{L}(A, B)$.

Theorem 3.10 *Let $Z(t)$ be the least element of the complete lattice $\mathcal{L}(A, B)$. Then the following statements hold.*

- (i) *For any given subspace $\mathcal{V} \subseteq \mathcal{S}$ the orthogonal projector $\tilde{Z}(t)$ onto the set $\mathcal{V}_t + \operatorname{Im} Z(t)$ belongs to $\mathcal{L}(A, B)$.*
- (ii) *For any given orthogonal projector $\tilde{Z}(t)$ from $\mathcal{L}(A, B)$ there exists a subspace $\mathcal{V} \subseteq \mathcal{S}$ such that $\tilde{Z}(t)$ is the orthogonal projector onto the set $\mathcal{V}_t + \operatorname{Im} Z(t)$ for all $t \in [a, \infty)$.*

Proof The statements follow directly from Theorem 3.9, because in this case the principal filter $\mathcal{Z}$ of $\mathcal{L}(A, B)$ generated by $Z(t)$ is equal to the whole lattice $\mathcal{L}(A, B)$. □

In the next corollary we discuss the situation when the matrix $B(t) \equiv 0$ on $[a, \infty)$, i.e., we analyze the homogeneous linear differential equation

$$x' = A(t)\, x, \quad t \in [a, \infty). \tag{3.7}$$

We note that in this case every orthogonal projector $Z(t)$ which solves (3.2) on $[a, \infty)$ satisfies trivially condition (3.4), i.e., the sets $\mathcal{L}(A, 0)$ and $\mathcal{L}(A)$ coincide. In addition, the matrix $Z(t) \equiv 0$ on $[a, \infty)$ is the least element of $\mathcal{L}(A, 0) = \mathcal{L}(A)$, by Remark 3.5(i). In particular, we characterize the elements of the set $\mathcal{L}(A)$ in terms of subspaces of solutions of (3.7).

Corollary 3.11 *Let $Z(t)$ be an orthogonal projector defined on $[a, \infty)$. Then the following statements are equivalent.*

- (i) *The matrix $Z(t)$ belongs to $\mathcal{L}(A)$.*
- (ii) *There exists a subspace $\mathcal{V}$ of solutions of (3.7) such that the matrix $Z(t)$ is the orthogonal projector onto the set $\mathcal{V}_t$ for every $t \in [a, \infty)$.*

Proof Suppose that the orthogonal projector $Z(t)$ belongs to $\mathcal{L}(A)$, i.e., the matrix $Z(t)$ solves (3.2) on $[a, \infty)$. According to Theorem 3.10(ii) with $B := 0$, $Z := 0$, and $\tilde{Z} := Z$ there exists a subspace $\mathcal{V}$ of solutions of (3.7) such that $Z(t)$ is the orthogonal projector onto the set $\mathcal{V}_t$ for all $t \in [a, \infty)$. Conversely, let $\mathcal{V}$ be a subspace of solutions of (3.7) such that $\mathcal{V}_t = \operatorname{Im} Z(t)$ for every $t \in [a, \infty)$. By using Theorem 3.10(i) again with $B := 0$, $Z := 0$, and $\tilde{Z} := Z$ we obtain that the matrix $Z(t)$ belongs to the set $\mathcal{L}(A, 0) = \mathcal{L}(A)$, which completes the proof. □

Remark 3.12 We note that the orthogonal projectors $Z(t)$ belonging to $\mathcal{L}(A)$ can be also classified in terms of subspaces of solutions of the equation

$$x' = -A^T(t)\, x, \quad t \in [a, \infty). \tag{3.8}$$

More precisely, an orthogonal projector $Z(t)$ belongs to $\mathcal{L}(A)$ if and only if there exists a subspace $\mathcal{V}$ of solutions of (3.8) such that $\mathcal{V}_t = \operatorname{Im}\, [I - Z(t)] = \operatorname{Ker} Z(t)$ for all $t \in [a, \infty)$. This follows directly from Corollary 3.11 applied to Eqs. (3.3) and (3.8), and Remark 3.5(ii).

In the last result of this section we provide a complete classification of all orthogonal projectors $Z(t)$ which belong to the set $\mathcal{L}(A, B)$. More precisely, we show that such orthogonal projectors $Z(t)$ correspond to the subspaces of solutions of the equations

$$z' = -A^T(t)\,z, \quad B^T(t)\,z = 0, \quad t \in [a, \infty). \tag{3.9}$$

We denote by Λ the linear space of all solutions of (3.9). For a given subspace $\mathcal{V} \subseteq \Lambda$ the symbol $\mathcal{V}_t$ will have the same meaning as in (3.1).

Theorem 3.13 *Let $Z(t)$ be an orthogonal projector defined on $[a, \infty)$. Then the following statements are equivalent.*

(i) *The matrix $Z(t)$ belongs to the set $\mathcal{L}(A, B)$.*
(ii) *There exists a subspace $\mathcal{V} \subseteq \Lambda$ of solutions of (3.9) such that the matrix $I - Z(t)$ is the orthogonal projector on the set $\mathcal{V}_t$ for every $t \in [a, \infty)$.*

Proof Let $Z(t)$ be an orthogonal projector belonging to $\mathcal{L}(A, B)$, i.e., the matrix $Z(t)$ solves (3.2) on $[a, \infty)$ and satisfies (3.4). According to Remark 3.12 there exists a subspace $\mathcal{V}$ of solutions of (3.8) such that the matrix $I - Z(t)$ is the orthogonal projector onto the set $\mathcal{V}_t$ for all $t \in [a, \infty)$. Moreover, by taking the orthogonal complements in condition (3.4) we get the inclusion $\operatorname{Im}\,[I - Z(t)] \subseteq \operatorname{Ker} B^T(t)$ for all $t \in [a, \infty)$. This shows that every function $z \in \mathcal{V}$ satisfies for all $t \in [a, \infty)$ the relation $z(t) \in \operatorname{Im}\,[I - Z(t)]$ and consequently, the equality $B^T(t)\,z(t) = 0$. Therefore, the set $\mathcal{V}$ is a subspace in Λ, i.e., a subspace of solutions of (3.9), showing (ii). Conversely, let $\mathcal{V} \subseteq \Lambda$ be a given subspace and let $Z(t)$ be an orthogonal projector such that $I - Z(t)$ is the orthogonal projector onto the set $\mathcal{V}_t$ for every $t \in [a, \infty)$. Then the matrix $Z(t)$ solves (3.2) on $[a, \infty)$, by Remark 3.12. Finally, the second equation in (3.9) implies that $B^T(t)\,[I - Z(t)] = 0$ on $[a, \infty)$, i.e., condition (3.4) holds. Therefore, the orthogonal projector $Z(t)$ belongs to $\mathcal{L}(A, B)$ and the proof is complete. □

Remark 3.14 We note that the statement of Theorem 3.13 can be equivalently formulated as follows. The matrix $Z(t)$ belongs to $\mathcal{L}(A, B)$ if and only if there exists a subspace $\mathcal{V} \subseteq \Lambda$ of solutions of (3.9) such that $Z(t)$ is the orthogonal projector onto the set $\mathcal{V}_t^{\perp}$ for every $t \in [a, \infty)$. Here $\mathcal{V}_t^{\perp} := (\mathcal{V}_t)^{\perp}$ denotes the orthogonal complement of the subspace $\mathcal{V}_t$ in $\mathbb{R}^n$.

Remark 3.15 From Theorem 3.13 and Remark 3.14 with the choice $\mathcal{V} = \Lambda$ it follows that the orthogonal projector onto the subspace $\Lambda_t^{\perp}$, i.e., the matrix

$$R_\Lambda(t) := \mathcal{P}_{\Lambda_t^{\perp}}, \quad t \in [a, \infty), \tag{3.10}$$

belongs to $\mathcal{L}(A, B)$. In particular, the function $Z(t) = R_\Lambda(t)$ on $[a, \infty)$ is then the least element of the lattice $\mathcal{L}(A, B)$. Similarly, the orthogonal projector $Z(t) \equiv I$ on $[a, \infty)$ which corresponds to the subspace $\mathcal{V} = \{0\}$ in Theorem 3.13, is the greatest element of $\mathcal{L}(A, B)$. We also note that with the aid of the orthogonal projector $R_\Lambda(t)$ in (3.10) the set $\mathcal{L}(A, B)$ can be characterized as follows. For any given $\alpha \in [a, \infty)$ the lattice $\mathcal{L}(A, B)$ is isomorphic to the principal filter of the complete lattice of all subspaces in $\mathbb{R}^n$ generated by $\operatorname{Im} R_\Lambda(\alpha)$.

4 Theory of Genera of Conjoined Bases

In this section we develop the theory of genera of conjoined bases of (H), when the Legendre condition (1.1) is not assumed and/or this system is allowed to be oscillatory. The presented theory generalizes the results for a nonoscillatory system (H) in [20,22]. In particular, we introduce a new definition of genus of conjoined bases (Definition 4.3) and present important properties of such an object. We derive a criterion saying when two conjoined bases belong to the same genus (Theorem 4.5), show that every genus can be characterized by a unique orthogonal projector $R_{\mathcal{G}}(t)$ which satisfies the Riccati equation (1.2) (Theorems 4.7 and 4.8), and prove that the set of all genera forms a complete lattice with respect to a suitable ordering (Definition 4.10 and Theorem 4.14). We also classify the conjoined bases belonging to the minimal genus $\mathcal{G}_{\min}$ of the above complete lattice (Theorem 4.15).

Following Remark 3.15 we define the orthogonal projector

$$R_{\Lambda\infty}(t) := \mathcal{P}_{\mathcal{W}_t^{\perp}}, \quad \text{where} \quad \mathcal{W}_t := \Lambda_t[\alpha_\infty, \infty), \quad t \in [\alpha_\infty, \infty), \tag{4.1}$$

where the point α_∞ is determined in (2.7) and $\Lambda_t[\alpha_\infty, \infty)$ is defined by (2.5). Moreover, following the notation in Sect. 3 we denote by $\mathcal{L}(A, B)_\infty$ the set of all orthogonal projectors $Z(t)$ which solve Eq. (3.2) and satisfy the inclusion in (3.4) on the interval $[\alpha_\infty, \infty)$.

Remark 4.1 (i) From the second identity in (2.9) it follows that for any $\alpha \geq \alpha_\infty$ the matrix $R_{\Lambda\infty}(t)$ defined in (4.1) is the orthogonal projector onto the set $\big(\Lambda_t[\alpha, \infty)\big)^{\perp}$ on $[\alpha, \infty)$, i.e.,

$$R_{\Lambda\infty}(t) = \mathcal{P}_{\mathcal{U}_t^{\perp}}, \quad \text{where} \quad \mathcal{U}_t := \Lambda_t[\alpha, \infty), \quad t \in [\alpha, \infty). \tag{4.2}$$

(ii) We note that in agreement with Remark 3.15 the orthogonal projector $R_{\Lambda\infty}(t)$ is the least element of the complete lattice $\mathcal{L}(A, B)_\infty$.

In the following auxiliary result we derive a relation between conjoined bases of (H) and orthogonal projectors of the set $\mathcal{L}(A, B)_\infty$.

Lemma 4.2 *Let (X, U) be a conjoined basis of (H). For each $t \in [\alpha_\infty, \infty)$ denote by $Z(t)$ the orthogonal projector onto the subspace* $\operatorname{Im} X(t) + \operatorname{Im} R_{\Lambda\infty}(t)$. *Then the matrix $Z(t)$ belongs to the set $\mathcal{L}(A, B)_\infty$.*

Proof Let (X, U) and $Z(t)$ be as in the statement. For a given $c \in \mathbb{R}^n$ consider the vector solution $(x, u) := (Xc, Uc)$ of (H). Since the pair (x, u) solves the equation of motion (1.4), it is an admissible pair. Moreover, let $\mathcal{V}$ be the subspace of the first components of all such admissible pairs (x, u), that is, the subspace of all vector-valued functions x of the form $x = Xc$ for some $c \in \mathbb{R}^n$. In particular, the equality $\mathcal{V}_t = \operatorname{Im} X(t)$ holds for all $t \in [a, \infty)$. From Remark 4.1(ii) and Theorem 3.10(i) it then follows that the orthogonal projector $Z(t)$ belongs to $\mathcal{L}(A, B)_\infty$. ◻

In [20, Definition 6.3] we introduced a genus of conjoined bases for a nonoscillatory system (H). Below we extend this notion to general possibly oscillatory system (H). We can observe that the orthogonal projector $R_{\Lambda\infty}(t)$ defined in (4.1) plays a crucial role in this extension, see also the comments in Remark 4.17(i). Also, below we do not require the validity of the Legendre condition (1.1).

Definition 4.3 (*Genus of conjoined bases*) Let (X_1, U_1) and (X_2, U_2) be two conjoined bases of (H). We say that (X_1, U_1) and (X_2, U_2) *have the same genus* (or they *belong to the same genus*) if there exists $\alpha \in [\alpha_\infty, \infty)$ such that

$$\operatorname{Im} X_1(t) + \operatorname{Im} R_{\Lambda\infty}(t) = \operatorname{Im} X_2(t) + \operatorname{Im} R_{\Lambda\infty}(t), \quad t \in [\alpha, \infty),$$

where α_∞ is defined in (2.7).

Remark 4.4 From Definition 4.3 it follows that the relation "having (or belonging to) the same genus" is an equivalence relation on the set of all conjoined bases of (H). Therefore, there exists a partition of this set into disjoint classes of conjoined bases of (H) with the same genus. This allows to interpret each equivalence class $\mathcal{G}$ as a genus itself.

In the following result we provide a fundamental property of conjoined bases of (H) which belong to the same genus.

Theorem 4.5 *Let (X_1, U_1) and (X_2, U_2) be conjoined bases of* (H). *Then the following statements are equivalent.*

- (i) *The conjoined bases (X_1, U_1) and (X_2, U_2) belong to the same genus.*
- (ii) *The equality* $\operatorname{Im} X_1(t) + \operatorname{Im} R_{\Lambda\infty}(t) = \operatorname{Im} X_2(t) + \operatorname{Im} R_{\Lambda\infty}(t)$ *holds for every* $t \in [\alpha_\infty, \infty)$.
- (iii) *The equality* $\operatorname{Im} X_1(t) + \operatorname{Im} R_{\Lambda\infty}(t) = \operatorname{Im} X_2(t) + \operatorname{Im} R_{\Lambda\infty}(t)$ *holds for some* $t \in [\alpha_\infty, \infty)$.

Proof Let (X_1, U_1) and (X_2, U_2) be as in the theorem and let $Z_1(t)$ and $Z_2(t)$ be the orthogonal projectors in Lemma 4.2 associated to the functions $X_1(t)$ and $X_2(t)$ on $[\alpha_\infty, \infty)$. In particular, the matrices $Z_1(t)$ and $Z_2(t)$ solve the Riccati equation (3.2) on $[\alpha_\infty, \infty)$. If (X_1, U_1) and (X_2, U_2) belong to the same genus, then the equality $Z_1(t) = Z_2(t)$ holds on $[\alpha, \infty)$ for some $\alpha \geq \alpha_\infty$, by Definition 4.3 and by the uniqueness of orthogonal projectors. From Proposition 3.1 it then follows that $Z_1(t) = Z_2(t)$ for all $t \in [\alpha_\infty, \infty)$, showing (ii). Conversely, (ii) implies (i) trivially. Finally, by using Proposition 3.1 once more and the uniqueness of solutions of (3.2) we obtain the equivalence of statements (ii) and (iii), which completes the proof. □

Let $\mathcal{G}$ be a genus of conjoined bases of (H) and let (X, U) be a conjoined basis of (H) belonging to $\mathcal{G}$. The definition of genus and the results in Theorem 4.5 imply that for all $t \in [\alpha_\infty, \infty)$ the subspace $\operatorname{Im} X(t) + \operatorname{Im} R_{\Lambda\infty}(t)$ does not depend on the particular choice of such a conjoined basis (X, U). Therefore, the orthogonal projector onto $\operatorname{Im} X(t) + \operatorname{Im} R_{\Lambda\infty}(t)$, i.e., the matrix

$$R_{\mathcal{G}}(t) := \mathcal{P}_{\mathcal{V}_t}, \quad \text{where} \quad \mathcal{V}_t := \operatorname{Im} X(t) + \operatorname{Im} R_{\Lambda\infty}(t), \quad t \in [\alpha_\infty, \infty), \tag{4.3}$$

is uniquely determined for each genus $\mathcal{G}$.

Remark 4.6 We note that the definition of the matrix $R_{\mathcal{G}}(t)$ in (4.3) implies that for every conjoined basis (X, U) of (H) in the genus $\mathcal{G}$ the associated orthogonal projector $Z(t)$ in Lemma 4.2 satisfies $Z(t) = R_{\mathcal{G}}(t)$ on $[\alpha_\infty, \infty)$. In particular, this shows that the orthogonal projectors $R_{\mathcal{G}}(t)$ are elements of the complete lattice $\mathcal{L}(A, B)_\infty$.

The next two theorems provide a classification of all genera $\mathcal{G}$ of conjoined bases of (H) in terms of their associated orthogonal projectors $R_{\mathcal{G}}(t)$ in (4.3). More precisely, we show that the matrices $R_{\mathcal{G}}(t)$ solve the Riccati equation (1.2) on $[\alpha_\infty, \infty)$ and satisfy the inclusion $\operatorname{Im} R_{\Lambda\infty}(t) \subseteq \operatorname{Im} R_{\mathcal{G}}(t)$ for all $t \in [\alpha_\infty, \infty)$.

Theorem 4.7 *Let $\mathcal{G}$ be a genus of conjoined bases of* (H) *and let $R_{\mathcal{G}}(t)$ be the orthogonal projector defined in* (4.3). *Then the matrix $R_{\mathcal{G}}(t)$ is a solution of the Riccati equation* (1.2) *on $[\alpha_\infty, \infty)$ and the inclusion $\operatorname{Im} R_{\Lambda\infty}(t) \subseteq \operatorname{Im} R_{\mathcal{G}}(t)$ holds for every $t \in [\alpha_\infty, \infty)$.*

Proof From Remark 4.6 we know that the matrix $R_{\mathcal{G}}(t)$ belongs to the complete lattice $\mathcal{L}(A, B)_\infty$. Therefore, the function $R_{\mathcal{G}}(t)$ solves (1.2) on $[\alpha_\infty, \infty)$. Moreover, by Remark 4.1(ii) the matrix $R_{\Lambda\infty}(t)$ defined in (4.1) is the least element of $\mathcal{L}(A, B)_\infty$, which in turn yields the condition $\mathrm{Im}\, R_{\Lambda\infty}(t) \subseteq \mathrm{Im}\, R_{\mathcal{G}}(t)$ for all $t \in [\alpha_\infty, \infty)$. □

Theorem 4.8 *Let $\beta \in [\alpha_\infty, \infty)$ be fixed and let $R \in \mathbb{R}^{n\times n}$ be an orthogonal projector satisfying* $\mathrm{Im}\, R_{\Lambda\infty}(\beta) \subseteq \mathrm{Im}\, R$. *Then there exists a unique genus $\mathcal{G}$ of conjoined bases of* (H) *such that its corresponding orthogonal projector $R_{\mathcal{G}}(t)$ in* (4.3) *satisfies $R_{\mathcal{G}}(\beta) = R$.*

Proof Let R and β be as in the theorem and consider the solution (X, U) of (H) given by the initial conditions $X(\beta) = R - R_{\Lambda\infty}(\beta)$ and $U(\beta) = I - R + R_{\Lambda\infty}(\beta)$. First we will prove that (X, U) is a conjoined basis. By using the identities $R\, R_{\Lambda\infty}(\beta) = R_{\Lambda\infty}(\beta) = R_{\Lambda\infty}(\beta)\, R$ we obtain that the matrix $X^T(\beta)\, U(\beta) = [R - R_{\Lambda\infty}(\beta)]\, [I - R + R_{\Lambda\infty}(\beta)] = 0$ is symmetric. Moreover, the equalities $\mathrm{rank}\, [X^T(\beta),\ U^T(\beta)] = \mathrm{rank}\, [R - R_{\Lambda\infty}(\beta),\ I - R + R_{\Lambda\infty}(\beta)] = n$ hold, because $\mathrm{Ker}\, [R - R_{\Lambda\infty}(\beta)] \cap \mathrm{Ker}\, [I - R + R_{\Lambda\infty}(\beta)] = \{0\}$. This shows that (X, U) is a conjoined basis of (H). Let $\mathcal{G}$ be the genus of conjoined bases of (H) such that $(X, U) \in \mathcal{G}$ and let $R_{\mathcal{G}}(t)$ be its corresponding matrix defined in (4.3). The equality $\mathrm{Im}\, [R - R_{\Lambda\infty}(\beta)] = \mathrm{Im}\, R \cap \mathrm{Ker}\, R_{\Lambda\infty}(\beta)$ implies that $\mathrm{Im}\, [R - R_{\Lambda\infty}(\beta)] \cap \mathrm{Im}\, R_{\Lambda\infty}(\beta) = \{0\}$. Therefore, the subspace $\mathcal{V}_t$ in (4.3) satisfies

$$\mathcal{V}_\beta = \mathrm{Im}\, [R - R_{\Lambda\infty}(\beta)] \oplus \mathrm{Im}\, R_{\Lambda\infty}(\beta) = \big(\mathrm{Im}\, R \cap \mathrm{Ker}\, R_{\Lambda\infty}(\beta)\big) \oplus \mathrm{Im}\, R_{\Lambda\infty}(\beta) = \mathrm{Im}\, R.$$

On the other hand, the identity $\mathcal{V}_\beta = \mathrm{Im}\, R_{\mathcal{G}}(\beta)$ holds. Thus, we have that $\mathrm{Im}\, R_{\mathcal{G}}(\beta) = \mathrm{Im}\, R$, which by the uniqueness of the orthogonal projectors yields the equality $R_{\mathcal{G}}(\beta) = R$. Finally, since the matrix $R_{\mathcal{G}}(t)$ solves the Riccati equation (1.2) or (3.2) on $[\alpha_\infty, \infty)$ by Theorem 4.7 and since the solutions of (3.2) are unique by Proposition 3.1, it follows that the genus $\mathcal{G}$ is uniquely determined by $R_{\mathcal{G}}(t)$, and hence by the orthogonal projector R. □

Remark 4.9 Combining the results from Remark 4.1(ii) and Theorems 4.7 and 4.8 allows to strengthen the observation in Remark 4.6. Namely, the matrices $R_{\mathcal{G}}(t)$ in (4.3) associated to genera $\mathcal{G}$ of conjoined bases of (H) coincide with the elements of the complete lattice $\mathcal{L}(A, B)_\infty$. We also note that every such orthogonal projector $R_{\mathcal{G}}(t)$, as a solution of (1.2) on $[\alpha_\infty, \infty)$, has constant rank $r_{\mathcal{G}}$ on the whole interval $[\alpha_\infty, \infty)$. In this context, we may adopt for the number $r_{\mathcal{G}}$ the terminology *rank of the genus* $\mathcal{G}$ and write $\mathrm{rank}\, \mathcal{G} := r_{\mathcal{G}}$, compare also with [21, Remark 6.4]. In particular, $n - d_\infty \le \mathrm{rank}\, \mathcal{G} \le n$ holds.

Based on the results in Theorems 4.7 and 4.8 we can now introduce an ordering on the set of all genera of conjoined bases of (H) which corresponds to the ordering in the lattice $\mathcal{L}(A, B)_\infty$.

Definition 4.10 Let $\mathcal{G}$ and $\mathcal{H}$ be two genera of conjoined bases of (H) and let $R_{\mathcal{G}}(t)$ and $R_{\mathcal{H}}(t)$ be their corresponding orthogonal projectors in (4.3), respectively. We say that the genus $\mathcal{G}$ is *below* the genus $\mathcal{H}$ (or that the genus $\mathcal{H}$ is *above* the genus $\mathcal{G}$) and we write $\mathcal{G} \preceq \mathcal{H}$ if the inclusion $\mathrm{Im}\, R_{\mathcal{G}}(t) \subseteq \mathrm{Im}\, R_{\mathcal{H}}(t)$ holds for all $t \in [\alpha_\infty, \infty)$.

Remark 4.11 We note that the genera $\mathcal{G}$ and $\mathcal{H}$ satisfy $\mathcal{G} \preceq \mathcal{H}$ if and only if the inclusion $\mathrm{Im}\, R_{\mathcal{G}}(\beta) \subseteq \mathrm{Im}\, R_{\mathcal{H}}(\beta)$ holds for some $\beta \in [\alpha_\infty, \infty)$. This follows from Definition 4.10 and Proposition 3.2, because the matrices $R_{\mathcal{G}}(t)$ and $R_{\mathcal{H}}(t)$ are solutions of the Riccati equation (1.2) on $[\alpha_\infty, \infty)$, by Theorem 4.7.

By the symbol Γ we will denote the set of all genera of conjoined bases of (H).

Theorem 4.12 *The relation $\preceq$ from Definition 4.10 is an ordering on the set Γ.*

Proof The statement follows from the one-to-one correspondence in Remarks 4.9 and 4.11 between the genera $\mathcal{G}$ of conjoined bases of (H) and the orthogonal projectors $R_{\mathcal{G}}(t)$ of the ordered set $\mathcal{L}(A, B)_\infty$. □

Remark 4.13 (i) Based on Proposition 3.3 and Remark 3.5(i) we can describe explicitly the infimum $\mathcal{G} \wedge \mathcal{H}$ and the supremum $\mathcal{G} \vee \mathcal{H}$ of two genera $\mathcal{G}$ and $\mathcal{H}$ of the set Γ. More precisely, if $R_{\mathcal{G}}(t)$ and $R_{\mathcal{H}}(t)$ are the orthogonal projectors associated to the genera $\mathcal{G}$ and $\mathcal{H}$, then $\mathcal{G} \wedge \mathcal{H}$ is the genus of conjoined bases corresponding to the orthogonal projector onto the subspace $\operatorname{Im} R_{\mathcal{G}}(t) \cap \operatorname{Im} R_{\mathcal{H}}(t)$ on $[\alpha_\infty, \infty)$, and $\mathcal{G} \vee \mathcal{H}$ is the genus of conjoined bases corresponding to the orthogonal projector onto the subspace $\operatorname{Im} R_{\mathcal{G}}(t) + \operatorname{Im} R_{\mathcal{H}}(t)$ on $[\alpha_\infty, \infty)$. This shows that $(\Gamma, \preceq)$ is a lattice.

(ii) If the orthogonal projector $R_{\mathcal{G}}(t)$ satisfies $R_{\mathcal{G}}(t) = R_{\Lambda\infty}(t)$ on $[\alpha_\infty, \infty)$, then the genus $\mathcal{G} = \mathcal{G}_{\min}$ is called *minimal*, while if $R_{\mathcal{G}}(t) \equiv I$ on $[\alpha_\infty, \infty)$, then the genus $\mathcal{G} = \mathcal{G}_{\max}$ is called *maximal*. We remark that this terminology is in full agreement with the results in Remarks 3.15 and 4.1(ii).

Theorem 4.14 *The ordered set $(\Gamma, \preceq)$ is a complete lattice. In particular, the minimal genus $\mathcal{G}_{\min}$ is the least element of Γ with respect to the ordering $\preceq$, while the maximal genus $\mathcal{G}_{\max}$ is the greatest element of Γ with respect to $\preceq$.*

Proof The statement follows directly from Theorem 4.12 and Remarks 4.9 and 4.13, since the lattice $(\Gamma, \preceq)$ is isomorphic to the complete lattice $\mathcal{L}(A, B)_\infty$. □

In the following result we characterize the conjoined bases belonging to the minimal genus $\mathcal{G}_{\min}$. We also show that the principal solution at the point α for $\alpha \in [\alpha_\infty, \infty)$ belongs to the minimal genus $\mathcal{G}_{\min}$. This statement generalizes [22, Proposition 4.7] to a possibly oscillatory system (H).

Theorem 4.15 *Let (X, U) be a conjoined basis of (H). Then (X, U) belongs to the minimal genus $\mathcal{G}_{\min}$ if and only if the inclusion $\operatorname{Im} X(t) \subseteq \operatorname{Im} R_{\Lambda\infty}(t)$ holds for some (and hence for every) $t \in [\alpha_\infty, \infty)$. In particular, for every $\alpha \geq \alpha_\infty$ the principal solution $(\hat{X}_\alpha, \hat{U}_\alpha)$ at the point α belongs to $\mathcal{G}_{\min}$.*

Proof Let $\mathcal{G}$ be the genus of conjoined bases of (H) such that $(X, U) \in \mathcal{G}$ and let $R_{\mathcal{G}}(t)$ be its representing orthogonal projector in (4.3). In particular, the inclusions $\operatorname{Im} X(t) \subseteq \operatorname{Im} R_{\mathcal{G}}(t)$ and $\operatorname{Im} R_{\Lambda\infty}(t) \subseteq \operatorname{Im} R_{\mathcal{G}}(t)$ hold on $[\alpha_\infty, \infty)$. If $\mathcal{G}$ is equal to the minimal genus $\mathcal{G}_{\min}$, then $R_{\mathcal{G}}(t) = R_{\Lambda\infty}(t)$ for all $t \in [\alpha_\infty, \infty)$, by Remark 4.13(ii), and in turn we obtain that $\operatorname{Im} X(t) \subseteq \operatorname{Im} R_{\mathcal{G}}(t) = \operatorname{Im} R_{\Lambda\infty}(t)$ for every $t \in [\alpha_\infty, \infty)$. Conversely, let $\beta \in [\alpha_\infty, \infty)$ be fixed and suppose that the inclusion $\operatorname{Im} X(\beta) \subseteq \operatorname{Im} R_{\Lambda\infty}(\beta)$ holds. The matrix $R_{\mathcal{G}}(\beta)$ then satisfies the equalities $\operatorname{Im} R_{\mathcal{G}}(\beta) = \operatorname{Im} X(\beta) + \operatorname{Im} R_{\Lambda\infty}(\beta) = \operatorname{Im} R_{\Lambda\infty}(\beta)$, which by the uniqueness of orthogonal projectors means that $R_{\mathcal{G}}(\beta) = R_{\Lambda\infty}(\beta)$. Therefore, the genus $\mathcal{G}$ is equal to $\mathcal{G}_{\min}$, by Theorem 4.8 and Remark 4.13(ii). Finally, if $(\hat{X}_\alpha, \hat{U}_\alpha)$ is the principal solution of (H) at some point $\alpha \in [\alpha_\infty, \infty)$, then $\hat{X}_\alpha(\alpha) = 0$ and hence, the inclusion $\operatorname{Im} \hat{X}_\alpha(\alpha) \subseteq \operatorname{Im} R_{\Lambda\infty}(\alpha)$ holds. This shows by the first part that $(\hat{X}_\alpha, \hat{U}_\alpha)$ belongs to the minimal genus $\mathcal{G}_{\min}$. □

In the next result we derive important properties of nonoscillatory conjoined bases from a given genus $\mathcal{G}$.

Proposition 4.16 *Let $\mathcal{G}$ be a genus of conjoined basis of (H) with the corresponding orthogonal projector $R_{\mathcal{G}}(t)$ in (4.3). Moreover, let (X, U) be a conjoined basis of (H) with constant kernel on $[\alpha, \infty) \subseteq [\alpha_\infty, \infty)$ such that (X, U) belongs to $\mathcal{G}$ and let $R(t)$ be the matrix defined in (2.1). Then the equality $R_{\mathcal{G}}(t) = R(t)$ holds for all $t \in [\alpha, \infty)$.*

Proof Let P be the constant orthogonal projector in (2.2) associated to (X, U) and put $(\tilde{X}, \tilde{U}) := \big(X(I-P),\ U(I-P)\big)$. By using the identity $X(t)\,P = X(t)$ for every $t \in [\alpha, \infty)$ we obtain that for any $c \in \mathbb{R}^n$ the pair $(\tilde{X}c, \tilde{U}c)$ is a vector solution of (H) satisfying $\tilde{X}(t)\,c \equiv 0$ on $[\alpha, \infty)$. Therefore, the function $\tilde{U}c \in \Lambda[\alpha, \infty)$. In particular, this means that the inclusion $\mathrm{Im}\,[U(t)\,(I-P)] \subseteq \Lambda_t[\alpha, \infty)$ holds for all $t \in [\alpha, \infty)$. Let $R_{\Lambda\infty}(t)$ be the orthogonal projector in (4.1) associated to the subspace $\Lambda_t[\alpha_\infty, \infty)$ on $[\alpha_\infty, \infty)$. By using the first equality in (2.4) and formula (4.2) in Remark 4.1(i) we have that

$$\mathrm{Ker}\,R(t) = \mathrm{Im}\,[U(t)\,(I-P)] \subseteq \Lambda_t[\alpha, \infty) = [\,\mathrm{Im}\,R_{\Lambda\infty}(t)\,]^{\perp} = \mathrm{Ker}\,R_{\Lambda\infty}(t) \tag{4.4}$$

for every $t \in [\alpha, \infty)$. In fact, taking the orthogonal complements implies that the relation in (4.4) is equivalent with the inclusion $\mathrm{Im}\,R_{\Lambda\infty}(t) \subseteq \mathrm{Im}\,R(t)$ on $[\alpha, \infty)$. Moreover, according to (4.3) the matrix $R_{\mathcal{G}}(t)$ satisfies $\mathrm{Im}\,R_{\mathcal{G}}(t) = \mathrm{Im}\,R(t) + \mathrm{Im}\,R_{\Lambda\infty}(t) = \mathrm{Im}\,R(t)$ on $[\alpha, \infty)$, which by the uniqueness of orthogonal projectors gives the equality $R_{\mathcal{G}}(t) = R(t)$ for all $t \in [\alpha, \infty)$. □

Remark 4.17 Assume that the Legendre condition (1.1) holds and that system (H) is nonoscillatory. By Proposition 2.1 this means that every conjoined basis of (H) is nonoscillatory. In particular, for any conjoined basis (X, U) of (H) there exists $\alpha \in [\alpha_\infty, \infty)$ such that (X, U) has constant kernel on $[\alpha, \infty)$. Moreover, let $\mathcal{G}$ be the genus of conjoined bases such that $(X, U) \in \mathcal{G}$ and let $R_{\mathcal{G}}(t)$ be its corresponding orthogonal projector in (4.3). Then the rank $r_{\mathcal{G}}$ of $\mathcal{G}$ defined in Remark 4.9 coincides with the rank r of any conjoined basis (X, U) of the genus $\mathcal{G}$ defined in (2.3). By Proposition 4.16 we have that $\mathrm{Im}\,X(t) = \mathrm{Im}\,R_{\mathcal{G}}(t)$ on $[\alpha, \infty)$. Therefore, from (4.3) and Definition 4.3 it follows that two conjoined bases (X_1, U_1) and (X_2, U_2) of (H) belong to the same genus if and only if there exists $\alpha \in [\alpha_\infty, \infty)$ such that the equality $\mathrm{Im}\,X_1(t) = \mathrm{Im}\,X_2(t)$ hold for every $t \in [\alpha, \infty)$. This observation shows that the concept given in Definition 4.3 generalizes the definition of genus of conjoined bases introduced in [20, Definition 6.3] for nonoscillatory system (H). We also note that the result about the structure of the set of all genera of conjoined bases presented in this section are in full agreement with the corresponding results in [22, Section 4].

In the following examples we illustrate the new theory of genera of conjoined bases of (H). We refer to [22, Section 5] for examples of nonoscillatory systems (H) with (1.1). Thus, in the first example we consider an oscillatory system (H) satisfying the Legendre condition (1.1).

Example 4.18 Let $n = 2$ and $A(t) \equiv 0$, $B(t) \equiv \mathrm{diag}\,\{1, 0\}$, and $C(t) \equiv \mathrm{diag}\,\{-1, 0\}$ on $[0, \infty)$. The principal solution $(\hat{X}_\alpha, \hat{U}_\alpha)$ at the point $\alpha = 0$ has the form

$$\big(\hat{X}_\alpha(t), \hat{U}_\alpha(t)\big) = \left(\begin{pmatrix} \sin t & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \cos t & 0 \\ 0 & 1 \end{pmatrix}\right). \tag{4.5}$$

Since $B(t) \geq 0$ on $[0, \infty)$ and the matrix $\hat{X}_\alpha(t)$ in (4.5) changes its kernel at each $t = k\pi$, $k \in \mathbb{N}$, system (H) is oscillatory, by Proposition 2.1. Moreover, we have $d_\infty = 1$, $\alpha_\infty = 0$, and $R_{\Lambda\infty}(t) \equiv \mathrm{diag}\,\{1, 0\}$ on $[0, \infty)$. Therefore, there exist only two genera of conjoined bases, i.e., the minimal genus $\mathcal{G}_{\min}$ with the corresponding orthogonal projector $R_{\mathcal{G}_{\min}}(t) =$

$R_{\Lambda\infty}(t)$ on $[0,\infty)$ and the maximal genus $\mathcal{G}_{\max}$ with the corresponding orthogonal projector $R_{\mathcal{G}_{\max}}(t) \equiv I$ on $[0,\infty)$. In particular, by (4.5) we have the identity

$$\operatorname{Im} \hat{X}_\alpha(t) + \operatorname{Im} R_{\Lambda\infty}(t) = \operatorname{Im} R_{\Lambda\infty}(t) = \operatorname{Im} R_{\mathcal{G}_{\min}}(t), \quad t \in [0,\infty),$$

and hence, the conjoined basis $(\hat{X}_\alpha, \hat{U}_\alpha)$ belongs to the minimal genus $\mathcal{G}_{\min}$, as we also claim in Theorem 4.15. On the other hand, the conjoined basis (X, U) of the form

$$\big(X(t), U(t)\big) = \left(\begin{pmatrix} \cos t & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -\sin t & 0 \\ 0 & 0 \end{pmatrix}\right) \tag{4.6}$$

is an element of the maximal genus $\mathcal{G}_{\max}$, because by (4.6) we have

$$\operatorname{Im} X(t) + \operatorname{Im} R_{\Lambda\infty}(t) = \mathbb{R}^2 = \operatorname{Im} R_{\mathcal{G}_{\max}}(t), \quad t \in [0,\infty).$$

In the second example we consider a system (H) which does not satisfy the Legendre condition (1.1). We note that this system is neither oscillatory nor nonoscillatory in the sense of Proposition 2.1.

Example 4.19 For $n = 3$ and $a = 0$ we consider system (H) with $A(t) = C(t) \equiv 0$ and

$$B(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e^t \\ 0 & e^t & 2e^{2t} \end{pmatrix} \quad \text{on } [0,\infty). \tag{4.7}$$

From (4.7) we can easily see that the matrix $B(t)$ is indefinite for all $t \in [0,\infty)$ and hence, the Legendre condition (1.1) does not hold. Moreover, we have $d_\infty = 1$, $\alpha_\infty = 0$, and $R_{\Lambda\infty}(t) \equiv \operatorname{diag}\{0, 1, 1\}$ on $[0,\infty)$. Therefore, system (H) is abnormal and it possesses only two genera of conjoined bases. Namely, there is the minimal genus $\mathcal{G} = \mathcal{G}_{\min}$ represented by the orthogonal projector $R_{\mathcal{G}}(t) = R_{\Lambda\infty}(t)$ on $[0,\infty)$, which contains for example the conjoined bases

$$\begin{aligned} \big(X_1(t), U_1(t)\big) &= \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e^t \\ 0 & e^t & e^{2t} \end{pmatrix}, I\right), \\ \big(X_2(t), U_2(t)\big) &= \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & e^t \\ 0 & e^t & e^{2t} \end{pmatrix}, I\right), \quad t \in [0,\infty), \end{aligned}$$

and there is the maximal genus $\mathcal{G} = \mathcal{G}_{\max}$ represented by the orthogonal projector $R_{\mathcal{G}}(t) \equiv I$ on $[0,\infty)$, which contains for example the conjoined bases

$$\big(X_3(t), U_3(t)\big) = (I, 0), \quad \big(X_4(t), U_4(t)\big) = \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^t \\ 0 & e^t & e^{2t} \end{pmatrix}, I\right), \quad t \in [0,\infty).$$

We note that the conjoined bases (X_1, U_1) and (X_3, U_3) have constant kernel on $[0,\infty)$ with the corresponding constant projectors $P_1 = \operatorname{diag}\{0, 1, 1\}$ and $P_2 = I$ in (2.2). On the other hand, the conjoined bases (X_2, U_2) and (X_4, U_4) do not have constant kernel on any nondegenerate subinterval in $[0,\infty)$ and their associated orthogonal projectors $P_2(t)$ and $P_4(t)$ in (2.1) are

$$P_2(t) = \frac{1}{e^{2t}+1}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & e^t \\ 0 & e^t & e^{2t} \end{pmatrix}, \quad P_4(t) = \frac{1}{e^{2t}+1}\begin{pmatrix} e^{2t}+1 & 0 & 0 \\ 0 & 1 & e^t \\ 0 & e^t & e^{2t} \end{pmatrix}, \quad t \in [0,\infty),$$

respectively. In particular, by using the terminology of Sect. 2 the conjoined bases (X_1, U_1) and (X_3, U_3) are both nonoscillatory on $[0, \infty)$, while the conjoined bases (X_2, U_2) and (X_4, U_4) are both oscillatory on $[0, \infty)$. This observation shows that in the absence of the Legendre condition (1.1) the statement of Proposition 2.1 does not hold, i.e., in this case it is not possible to classify system (H) as nonoscillatory or oscillatory. In spite of that both conjoined bases (X_2, U_2) and (X_4, U_4) have constant rank on the interval $[0, \infty)$. Namely, in agreement with (2.3) we have $r_2 = \operatorname{rank} X_2(t) \equiv 1$ and $r_4 = \operatorname{rank} X_4(t) \equiv 2$ on $[0, \infty)$.

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APPENDIX B

Paper by Šepitka (DCDS 2019)

This paper entitled "Riccati equations for linear Hamiltonian systems without controllability condition" appeared in the journal Discrete and Continuous Dynamical Systems, 39 (2019), no. 4, 1685–1730, see item [77] in the bibliography. This paper is dedicated to the memory of Professor Russell A. Johnson.

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RICCATI EQUATIONS FOR LINEAR HAMILTONIAN SYSTEMS WITHOUT CONTROLLABILITY CONDITION

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Dedicated to the memory of Professor Russell A. Johnson.

(Communicated by Alberto Bressan)

Abstract. In this paper we develop new theory of Riccati matrix differential equations for linear Hamiltonian systems, which do not require any controllability assumption. When the system is nonoscillatory, it is known from our previous work that conjoined bases of the system with eventually the same image form a special structure called a genus. We show that for every such a genus there is an associated Riccati equation. We study the properties of symmetric solutions of these Riccati equations and their connection with conjoined bases of the system. For a given genus, we pay a special attention to distinguished solutions at infinity of the associated Riccati equation and their relationship with the principal solutions at infinity of the system in the considered genus. We show the uniqueness of the distinguished solution at infinity of the Riccati equation corresponding to the minimal genus. This study essentially extends and completes the work of W. T. Reid (1964, 1972), W. A. Coppel (1971), P. Hartman (1964), W. Kratz (1995), and other authors who considered the Riccati equation and its distinguished solution at infinity for invertible conjoined bases, i.e., for the maximal genus in our setting.

1. **Introduction.** Riccati differential equations for self-adjoint linear differential systems play fundamental role in mathematical research as well as in applications. Specifically, if $n \in \mathbb{N}$ is a given dimension and $A, B, C : [a, \infty) \to \mathbb{R}^{n\times n}$ are given piecewise continuous matrix-valued functions such that $B(t)$ and $C(t)$ are symmetric, the Riccati matrix differential equation

$$Q' + Q\,A(t) + A^T(t)\,Q + Q\,B(t)\,Q - C(t) = 0 \tag{R}$$

is associated with the linear Hamiltonian system

$$x' = A(t)\,x + B(t)\,u, \quad u' = C(t)\,x - A^T(t)\,u, \tag{H}$$

see [7, 9, 17, 22, 23, 24]. It is known that under the Legendre condition

$$B(t) \geq 0 \quad \text{for all } t \in [a, \infty) \tag{1.1}$$

the Riccati equation (R) has many applications in various disciplines, for example in the oscillation and spectral theory [2, 7, 17, 22, 23, 24], filtering and prediction theory [16, 23], calculus of variations and optimal control theory [1, 3, 8, 12, 10,

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19, 34, 35, 37, 38, 39], systems theory and control [14, 15], and others (engineering, etc.).

In [20], Reid showed that when system (H) is completely controllable and nonoscillatory, the Riccati equation (R) has the so-called distinguished solution $\hat{Q}(t)$ at infinity, which is the smallest symmetric solution of (R) existing on an interval $[\alpha, \infty)$ for some $\alpha \geq a$. In the subsequent paper [21], Reid derived the minimality of the distinguished solution of (R) at infinity also for a noncontrollable system (H) by considering invertible principal solutions $(\hat{X}, \hat{U})$ of (H) at infinity. Recently, the author and Šimon Hilscher developed the theory of principal solutions at infinity for a general nonoscillatory and possibly abnormal system (H). We showed in [28, 29] the existence of principal solutions $(\hat{X}, \hat{U})$ at infinity with all ranks of $\hat{X}(t)$ in a specific range depending on the maximal order of abnormality d_∞ of (H), their classification and limit properties with antiprincipal solutions at infinity [30], and the geometric structure of the set of all conjoined bases [31]. In particular, conjoined bases of (H) with eventually the same image of the first component form a genus $\mathcal{G}$, which can be represented by an orthogonal projector $R_{\mathcal{G}}(t)$ satisfying the Riccati type matrix differential equation

$$R_{\mathcal{G}}' - A(t)\, R_{\mathcal{G}} - R_{\mathcal{G}}\, A^T(t) + R_{\mathcal{G}}\, [A(t) + A^T(t)]\, R_{\mathcal{G}} = 0. \tag{1.2}$$

This leads under (1.1) to a complete description of the set Γ of all genera of conjoined bases of a nonoscillatory system (H), being a complete lattice [31, Theorem 4.8]. This result was recently extended to a possibly oscillatory system (H) in [27, Theorem 4.14].

In this paper we continue in the above study of linear Hamiltonian system (H) by developing the corresponding theory of Riccati matrix differential equations. The presented approach and results are novel in three directions:

(i) we do not require any controllability assumption on system (H),

(ii) for every genus $\mathcal{G}$ we associate a Riccati equation

$$Q' + Q\, \mathcal{A}(t) + \mathcal{A}^T(t)\, Q + Q\, \mathcal{B}(t)\, Q - \mathcal{C}(t) = 0, \tag{$\mathcal{R}$}$$

where the coefficients $\mathcal{A}(t)$, $\mathcal{B}(t)$, and $\mathcal{C}(t)$ are given by

$$\left.\begin{aligned} \mathcal{A}(t) &:= A(t) R_{\mathcal{G}}(t) - A^T(t)\, [I - R_{\mathcal{G}}(t)], \\ \mathcal{B}(t) &:= B(t), \quad \mathcal{C}(t) := R_{\mathcal{G}}(t)\, C(t)\, R_{\mathcal{G}}(t), \end{aligned}\right\} \tag{1.3}$$

(iii) we show that every such a Riccati equation ($\mathcal{R}$) possesses a distinguished solution at infinity (defined in a suitable way), which corresponds to a principal solution of (H) at infinity from the genus $\mathcal{G}$.

More precisely, given a genus $\mathcal{G}$ of conjoined bases of (H), we show (Theorems 4.18 and 4.21) a fundamental connection between the symmetric solutions $Q(t)$ of ($\mathcal{R}$) on $[\alpha, \infty)$ with some $\alpha \geq a$ satisfying

$$\operatorname{Im} Q(t) \subseteq \operatorname{Im} R_{\mathcal{G}}(t), \quad t \in [\alpha, \infty), \tag{1.4}$$

and the conjoined bases (X, U) of (H) with constant kernel on $[\alpha, \infty)$, which belong to $\mathcal{G}$. We define (Definition 7.1) a distinguished solution $\hat{Q}(t)$ at infinity for each Riccati equation ($\mathcal{R}$), which corresponds to a principal solution $(\hat{X}, \hat{U})$ of (H) at infinity in the genus $\mathcal{G}$. We also prove (Theorem 7.16) that for every symmetric solution $Q(t)$ of ($\mathcal{R}$) on $[\alpha, \infty)$ with (1.4) there exists a distinguished solution $\hat{Q}(t)$

of ($\mathcal{R}$) satisfying the inequality

$$Q(t) \geq \hat{Q}(t) \quad \text{on } [\alpha, \infty). \tag{1.5}$$

The above results are particularly important for the minimal genus $\mathcal{G} = \mathcal{G}_{\min}$, which is formed by the conjoined bases (X, U) of (H) with minimal possible rank of the matrix $X(t)$, i.e., with $\operatorname{rank} X(t) = n - d_\infty$ on $[\alpha, \infty)$. In this case the associated distinguished solution $\hat{Q}_{\min}(t)$ at infinity is unique and minimal among all symmetric solutions $Q(t)$ of ($\mathcal{R}$) satisfying (1.4). This latter situation generalizes the classical controllable results of Reid and Coppel [7, 20, 22], since in this case $d_\infty = 0$ and the orthogonal projector $R_{\mathcal{G}}(t) \equiv I$ on $[a, \infty)$, so that the Riccati equation ($\mathcal{R}$) reduces to (R). We note that the original results by Reid [21, 23] for noncontrollable system (H) and Riccati equation (R) correspond in our new theory to the maximal genus $\mathcal{G} = \mathcal{G}_{\max}$ of conjoined bases (X, U) with eventually invertible matrix $X(t)$, i.e., to $R_{\mathcal{G}}(t) \equiv I$ on $[a, \infty)$. Therefore, the present study can be regarded as a generalization and completion of the theory of the Riccati equations (R) for completely controllable systems (H) using the minimal genus $\mathcal{G} = \mathcal{G}_{\min}$, as well as the noncontrollable systems (H) using the maximal genus $\mathcal{G} = \mathcal{G}_{\max}$.

Among other new results in this paper (Theorem 6.3 and Corollary 6.4) we mention a connection of the symmetric solutions $Q(t)$ of ($\mathcal{R}$) with the implicit Riccati equation

$$R_{\mathcal{G}}(t)\, [Q' + Q\, A(t) + A^T(t)\, Q + Q\, B(t)\, Q - C(t)]\, R_{\mathcal{G}}(t) = 0. \tag{1.6}$$

Such implicit Riccati equations occur in the study of nonnegative quadratic functional associated with system (H), see [13, Section 6].

The study of the Riccati equations in the context of the present paper is also motivated by several situations in the literature, which are equivalent to using the Riccati matrix differential equation for an uncontrollable linear Hamiltonian system. For example, in [35, pg. 886], [1, pp. 621–622], [11, Sections 4 and 6], and [12, pp. 17–18] the authors use a cascade system of three differential equations for the investigation of calculus of variations or optimal control problems with variable endpoints – the Riccati equation (R), a linear differential equation, and an integrator. These three differential equations are together equivalent to a Riccati equation in dimension $2n$, which corresponds to an uncontrollable system (H) in dimension $4n$. This connection is discusses in details in [11, Remark 6.3].

The results of this paper open new directions in the theory of Riccati matrix differential equations associated with general uncontrollable linear Hamiltonian systems. They demonstrate that, as in the completely controllable case, distinguished solutions at infinity play a prominent role in the structure of the space of symmetric solutions of ($\mathcal{R}$). Moreover, the intimate connection with the principal solutions of (H) at infinity points to effective applications of the distinguished solutions of ($\mathcal{R}$) at infinity in other fields of mathematics and engineering.

The paper is organized as follows. In Section 2 we display the notation and preliminary results about system (H) and its solutions. In Section 3 we present properties of principal solutions of (H) at infinity and recall the concept of a genus of conjoined bases of (H). In Section 4 we develop the theory of Riccati differential equations for a given genus $\mathcal{G}$. In Section 5 we study inequalities for Riccati type quotients associated with the Riccati equation ($\mathcal{R}$). In Section 6 we analyze the relationship between the two Riccati equations ($\mathcal{R}$) and (1.6). In Section 7 we define the notion of a distinguished solution of ($\mathcal{R}$) at infinity and study its minimality properties. Finally, in Section 8 we provide examples illustrating our new theory.

2. **Preliminaries about linear Hamiltonian systems.** In this section we review some recent results about linear Hamiltonian systems (H) from [18, 33, 28, 29, 30, 31]. For a general theory of these systems we refer to [7, 17, 22]. By a matrix solution of (H) we mean a pair of functions (X, U) such that $X, U : [a, \infty) \to \mathbb{R}^{n\times n}$ are piecewise continuously differentiable (C^1_p) and satisfy system (H) on $[a, \infty)$. In order to shorten the notation and the calculations, we sometimes suppress the argument t in the solutions. For any two matrix solutions (X_1, U_1) and (X_2, U_2) of (H) their Wronskian $X_1^T U_2 - U_1^T X_2$ is constant on $[a, \infty)$. A solution (X, U) of (H) is called a *conjoined basis* if $\operatorname{rank}(X^T(t),\, U^T(t))^T = n$ and $X^T(t)\, U(t)$ is symmetric at some and hence at any $t \in [a, \infty)$. The principal solution $(\hat{X}_\alpha, \hat{U}_\alpha)$ at the point $\alpha \in [a, \infty)$ is defined by the initial conditions $\hat{X}_\alpha(\alpha) = 0$ and $\hat{U}_\alpha(\alpha) = I$. By [17, Corollary 3.3.9], a given conjoined basis (X, U) can be completed to a fundamental system of (H) by another conjoined basis $(\bar{X}, \bar{U})$. In addition, the conjoined basis $(\bar{X}, \bar{U})$ can be chosen so that (X, U) and $(\bar{X}, \bar{U})$ are *normalized*, i.e., we have

$$X^T \bar{U} - U^T \bar{X} = I. \tag{2.1}$$

The oscillation of conjoined bases of (H) is defined via the concept of proper focal points, see [36, Definition 1.1]. However, this concept will not be explicitly needed in this paper. By [33, Definition 2.1], a conjoined basis (X, U) of (H) is called *nonoscillatory* if there exists $\alpha \in [a, \infty)$ such that $\operatorname{Ker} X(t)$ is constant on $[\alpha, \infty)$. The main result of [33] then describes the nonoscillatory behavior of system (H), see Proposition 2.1 below. Based on this result we say that system (H) is *nonoscillatory* if one and hence all conjoined bases of (H) are nonoscillatory.

Proposition 2.1. *Assume that the Legendre condition* (1.1) *holds. Then there exists a nonoscillatory conjoined basis of* (H) *if and only if every conjoined basis of* (H) *is nonoscillatory.*

For a subspace $\mathcal{V} \subseteq \mathbb{R}^n$ we denote by $\mathcal{P}_\mathcal{V}$ the orthogonal projector onto $\mathcal{V}$. That is, $\mathcal{P}_\mathcal{V}$ is a symmetric and idempotent $n\times n$ matrix such that $\operatorname{Im}\mathcal{P}_\mathcal{V} = \mathcal{V} = \operatorname{Ker}(I - \mathcal{P}_\mathcal{V})$ and $\operatorname{Ker}\mathcal{P}_\mathcal{V} = \mathcal{V}^\perp = \operatorname{Im}(I - \mathcal{P}_\mathcal{V})$. Orthogonal projectors can be constructed by using the Moore–Penrose pseudoinverse. More precisely, for a given matrix $M \in \mathbb{R}^{n\times n}$ and its pseudoinverse $M^\dagger$ the matrix $MM^\dagger$ is the orthogonal projector onto $\operatorname{Im} M$, and the matrix $M^\dagger M$ is the orthogonal projector onto $\operatorname{Im} M^\dagger = \operatorname{Im} M^T$. Moreover, $\operatorname{rank} M = \operatorname{rank} MM^\dagger = \operatorname{rank} M^\dagger M$ and $\operatorname{Ker}(MN) = \operatorname{Ker}(M^\dagger MN)$ for any matrices $M, N \in \mathbb{R}^{n\times n}$. For the theory of pseudoinverse matrices we refer to [4], [5, Chapter 6], and [6, Section 1.4]. In particular, we will need the following results on the differentiability of the Moore–Penrose pseudoinverse of a matrix-valued function $M(t)$.

Remark 2.2. By [6, Theorems 10.5.1 and 10.5.3], for a differentiable matrix-valued function $M(t)$ on an interval $[\alpha, \infty)$ its Moore–Penrose pseudoinverse $M^\dagger(t)$ is differentiable on $[\alpha, \infty)$ if and only if $\operatorname{rank} M(t)$ is constant on $[\alpha, \infty)$. In this case (suppressing the argument t)

$$(M^\dagger)' = -M^\dagger M' M^\dagger + (I - M^\dagger M)\,(M')^T M^{\dagger T} M^\dagger + M^\dagger M^{\dagger T} (M')^T (I - MM^\dagger) \tag{2.2}$$

on $[\alpha,\infty)$, see also [28, Remark 2.3]. Moreover, when $\operatorname{Ker} M(t)$ is constant on $[\alpha,\infty)$, then we have $\operatorname{Ker} M(t) \subseteq \operatorname{Ker} M'(t)$ on $[\alpha,\infty)$ and hence (2.2) reduces to

$$(M^\dagger)'(t) = -M^\dagger(t)\, M'(t)\, M^\dagger(t) + M^\dagger(t)\, M^{\dagger T}(t)\, (M')^T(t)\, [I - M(t)\, M^\dagger(t)] \quad (2.3)$$

for every $t \in [\alpha,\infty)$. In particular, when the matrix $M(t)$ is symmetric and $\operatorname{Ker} M(t)$ is constant on $[\alpha,\infty)$, then (2.3) yields the standard formula

$$(M^\dagger)'(t) = -M^\dagger(t)\, M'(t)\, M^\dagger(t), \quad t \in [\alpha,\infty).$$

In the rest of this section (except of Theorem 2.9) we present known properties of conjoined bases of (H) with the corresponding references to the literature. Given a conjoined basis (X,U) of (H), by the *kernel*, *image*, and *rank* of (X,U) we mean the kernel, image, and rank of the component X. On the interval $[a,\infty)$ we define the orthogonal projectors onto the subspaces $\operatorname{Im} X^T(t)$ and $\operatorname{Im} X(t)$ by

$$P(t) := \mathcal{P}_{\operatorname{Im} X^T(t)} = X^\dagger(t)\, X(t), \quad R(t) := \mathcal{P}_{\operatorname{Im} X(t)} = X(t)\, X^\dagger(t). \quad (2.4)$$

If (X,U) has constant kernel on $[\alpha,\infty) \subseteq [a,\infty)$, then by (2.4) the function $P(t)$ is constant on $[\alpha,\infty)$ and we set

$$P := P(t) \quad \text{on } [\alpha,\infty). \quad (2.5)$$

In this case (X,U) has constant rank r on $[\alpha,\infty)$ with

$$r := \operatorname{rank} X(t) = \operatorname{rank} P = \operatorname{rank} R(t) \quad \text{on } [\alpha,\infty), \quad (2.6)$$

and hence it follows from Remark 2.2 that the function $X^\dagger \in \mathrm{C}^1_{\mathrm{p}}$ on $[\alpha,\infty)$. Consequently, the *Riccati quotient*

$$Q(t) := X(t)\, X^\dagger(t)\, U(t)\, X^\dagger(t) = R(t)\, U(t)\, X^\dagger(t), \quad t \in [\alpha,\infty), \quad (2.7)$$

is piecewise continuously differentiable on $[\alpha,\infty)$ as well. In addition, by [25, pg. 24] the matrix $Q(t)$ is symmetric and satisfies on $[\alpha,\infty)$ the properties (suppressing the argument t)

$$X^T Q X = X^T U, \quad \operatorname{Im} Q \subseteq \operatorname{Im} R, \quad QX = RU. \quad (2.8)$$

The next statement is proven in [28, Theorem 4.2 and Equation (4.8)]. We observe that the Legendre condition (1.1) is not needed in this case.

Proposition 2.3. *Let (X,U) be a conjoined basis of (H) with constant kernel on the interval $[\alpha,\infty) \subseteq [a,\infty)$ and let P, $R(t)$, and $Q(t)$ be the corresponding matrices in (2.5), (2.4), and (2.7). Then the equalities*

$$\operatorname{Im}[U(t)\,(I-P)] = \operatorname{Ker} R(t), \qquad B(t) = R(t)\, B(t) = B(t)\, R(t) \quad (2.9)$$

hold for all $t \in [\alpha,\infty)$. Moreover, the matrix $R(t)$ solves the Riccati equation (1.2) on $[\alpha,\infty)$, while $X^\dagger$ satisfies on $[\alpha,\infty)$ the formula

$$(X^\dagger)' = X^\dagger A^T (I-R) - X^\dagger A R - X^\dagger B Q. \quad (2.10)$$

Following [28, Section 4], with any conjoined basis (X,U) of (H) with constant kernel on $[\alpha,\infty)$ we associate the *S-matrix* as the matrix-valued function

$$S_\alpha(t) := \int_\alpha^t X^\dagger(s)\, B(s)\, X^{\dagger T}(s)\, \mathrm{d}s, \quad t \in [\alpha,\infty). \quad (2.11)$$

Under (1.1), the matrix $S_\alpha(t)$ is symmetric, nonnegative definite, $S_\alpha \in \mathrm{C}^1_{\mathrm{p}}$ on $[\alpha,\infty)$, and by [28, Theorem 4.2] the set $\operatorname{Im} S_\alpha(t)$ is nondecreasing on $[\alpha,\infty)$ and hence eventually constant with $\operatorname{Im} S_\alpha(t) \subseteq \operatorname{Im} P$. By the symmetry of $S_\alpha(t)$, the set $\operatorname{Ker} S_\alpha(t)$ is nonincreasing on $[\alpha,\infty)$ and hence eventually constant with $\operatorname{Ker} P \subseteq \operatorname{Ker} S_\alpha(t)$.

This implies that the orthogonal projector onto the set $\operatorname{Im} S_\alpha(t)$ is eventually constant and we write

$$\left.\begin{aligned} P_{\mathcal{S}_\alpha}(t) &:= \mathcal{P}_{\operatorname{Im} S_\alpha(t)} = S_\alpha(t)\, S_\alpha^\dagger(t) = S_\alpha^\dagger(t)\, S_\alpha(t), \\ P_{\mathcal{S}_\alpha\infty} &:= P_{\mathcal{S}_\alpha}(t) \quad \text{for } t \to \infty. \end{aligned}\right\} \tag{2.12}$$

In addition, on $[\alpha, \infty)$ we have the inclusions

$$\operatorname{Im} S_\alpha(t) = \operatorname{Im} P_{\mathcal{S}_\alpha}(t) \subseteq \operatorname{Im} P_{\mathcal{S}_\alpha\infty} \subseteq \operatorname{Im} P.$$

The main properties of the function $S_\alpha(t)$ are summarized in the following statement, which follows from the definition of $S_\alpha(t)$ in (2.11), Remark 2.2, and (1.1), see also [28, Theorem 6.1].

Proposition 2.4. *Assume* (1.1). *Let (X, U) be a conjoined basis of* (H) *with constant kernel on $[\alpha, \infty)$ and let $S_\alpha(t)$ be the corresponding matrix defined in* (2.11). *Then the matrix function $S_\alpha(t)$ is nondecreasing on $[\alpha, \infty)$. Moreover, if $S_\alpha(t)$ has constant kernel on a subinterval $\mathcal{I} \subseteq [\alpha, \infty)$, then $S_\alpha^\dagger \in \mathrm{C}^1_{\mathrm{p}}(\mathcal{I})$ and $S_\alpha^\dagger(t)$ is nonincreasing on $\mathcal{I}$. In particular, if $S_\alpha(t)$ has constant kernel on $\mathcal{I} = [\beta, \infty)$, then the limit of $S_\alpha^\dagger(t)$ as $t \to \infty$ exists.*

Remark 2.5. Under (1.1), the results in Proposition 2.4 and the properties of the matrix function $S_\alpha(t)$ discussed above imply that for every conjoined basis (X, U) of (H) with constant kernel on an interval $[\alpha, \infty)$ the limit

$$T_\alpha := \lim_{t\to\infty} S_\alpha^\dagger(t) \tag{2.13}$$

is well defined and it is referred to as the T*-matrix* corresponding to the conjoined basis (X, U) on $[\alpha, \infty)$. Moreover, the matrix T_α is symmetric, nonnegative definite, and $\operatorname{Im} T_\alpha \subseteq \operatorname{Im} P_{\mathcal{S}_\alpha\infty}$ by (2.12) and $\operatorname{Im} S_\alpha^\dagger(t) = \operatorname{Im} S_\alpha^T(t) = \operatorname{Im} S_\alpha(t)$ on $[\alpha, \infty)$.

Remark 2.6. The matrix $S_\alpha(t)$ is intimately connected with a certain class of conjoined bases of (H) which are normalized with (X, U). As we showed in [28, Theorem 4.4], for a given conjoined basis (X, U) with constant kernel on $[\alpha, \infty)$ there exists a conjoined basis $(\bar{X}, \bar{U})$ of (H) such that (X, U) and $(\bar{X}, \bar{U})$ are normalized, i.e., (2.1) holds, and

$$X^\dagger(\alpha)\, \bar{X}(\alpha) = 0. \tag{2.14}$$

The matrices $\bar{X}(t)$, $\bar{X}(t)\, P$, and $\bar{U}(t)\, P$ are uniquely determined by (X, U) on the interval $[\alpha, \infty)$ and

$$\left.\begin{aligned} &\bar{X}(t)\, P = X(t)\, S_\alpha(t), \\ &\bar{U}(t)\, P = U(t)\, S_\alpha(t) + X^{\dagger T}(t) + U(t)\, (I - P)\, \bar{X}^T(t)\, X^{\dagger T}(t) \end{aligned}\right\} \tag{2.15}$$

for every $t \in [\alpha, \infty)$, where P is given in (2.5), see [28, Remark 4.5.(ii)]. We also note that according to [28, Theorem 5.2] the matrix $S_\alpha(t)$ satisfies the identities

$$\hat{X}_\alpha(t) = X(t)\, S_\alpha(t)\, X^T(\alpha), \quad X^\dagger(t)\, \hat{X}_\alpha(t) = S_\alpha(t)\, X^T(\alpha), \quad t \in [\alpha, \infty), \tag{2.16}$$

where $(\hat{X}_\alpha, \hat{U}_\alpha)$ is the principal solution of (H) at the point α.

As it is common, see [23, Section 3] or [28, Section 5], we denote by $\Lambda[\alpha, \infty)$ the linear space of n-dimensional vector-valued functions $u \in \mathrm{C}^1_{\mathrm{p}}$, which satisfy the equations $u' = -A^T(t)\, u$ and $B(t)\, u = 0$ on $[\alpha, \infty)$. The functions $u \in \Lambda[\alpha, \infty)$ correspond to the vector solutions $(x \equiv 0, u)$ of system (H) on $[\alpha, \infty)$. The space $\Lambda[\alpha, \infty)$ is finite-dimensional with $d[\alpha, \infty) := \dim \Lambda[\alpha, \infty) \le n$. The number $d[\alpha, \infty)$ is called the *order of abnormality* of system (H) on the interval $[\alpha, \infty)$.

We remark that system (H) is said to be *normal* on $[\alpha,\infty)$ if $d[\alpha,\infty)=0$, while it is called *identically normal* (or *completely controllable*) on $[\alpha,\infty)$ if $d(\mathcal{I})=0$ for every nondegenerate subinterval $\mathcal{I}\subseteq[\alpha,\infty)$. Moreover, for a given $t\in[\alpha,\infty)$ we denote by $\Lambda_t[\alpha,\infty)$ the subspace in $\mathbb{R}^n$ of values of functions $u\in\Lambda[\alpha,\infty)$ at the point t, i.e.,

$$\Lambda_t[\alpha,\infty):=\{c\in\mathbb{R}^n,\ u(t)=c\ \text{ for some }\ u\in\Lambda[\alpha,\infty)\},\quad t\in[\alpha,\infty). \tag{2.17}$$

It is easy to see that $\dim\Lambda_t[\alpha,\infty)=d[\alpha,\infty)$ for all $t\in[\alpha,\infty)$. We note that the set $\Lambda[t,\infty)$ is nondecreasing in t on $[a,\infty)$ and hence it is eventually constant. This means that the integer-valued function $d[t,\infty)$ is nondecreasing, piecewise constant, and right-continuous on $[a,\infty)$. In particular, there exists the limit

$$d_\infty:=\lim_{t\to\infty}d[t,\infty)=\max_{t\in[a,\infty)}d[t,\infty),\quad 0\le d_\infty\le n, \tag{2.18}$$

which is called the *maximal order of abnormality* of (H). The monotonicity of the function $d[t,\infty)$ justifies the existence of the point $\alpha_\infty\in[a,\infty)$ such that

$$\alpha_\infty:=\min\{\alpha\in[a,\infty),\ d[\alpha,\infty)=d_\infty\}. \tag{2.19}$$

From (2.18) and (2.19) we then obtain that the subspace $\Lambda[\alpha_\infty,\infty)$ satisfies

$$\left.\begin{aligned}\Lambda[\alpha_\infty,\infty)=\lim\nolimits_{\alpha\to\infty}\Lambda[\alpha,\infty)=\max\nolimits_{\alpha\in[a,\infty)}\Lambda[\alpha,\infty),\\ \Lambda[\alpha,\infty)\equiv\Lambda[\alpha_\infty,\infty),\quad \alpha\in[\alpha_\infty,\infty).\end{aligned}\right\} \tag{2.20}$$

On the other hand, for any $\alpha\in[a,\infty)$ the subspace $\Lambda[\alpha,t]$ is nonincreasing in t on (α,∞) and hence it is eventually constant. In particular, the integer-valued function $d[\alpha,t]$ is nonincreasing, piecewise constant, and left-continuous on (α,∞), see also [28, Section 5]. Moreover, we get

$$d[\alpha,\infty)=\lim_{t\to\infty}d[\alpha,t]=\min_{t\in(\alpha,\infty)}d[\alpha,t], \tag{2.21}$$

$$\Lambda[\alpha,\infty)=\lim_{t\to\infty}\Lambda[\alpha,t]=\min_{t\in(\alpha,\infty)}\Lambda[\alpha,t] \tag{2.22}$$

for all $\alpha\in[a,\infty)$. For any such a point α the relation in (2.21) and the above properties of the function $d[\alpha,t]$ yield the existence of the point $\tau_{\alpha,\infty}$ in the interval $[\alpha,\infty)$ such that

$$\tau_{\alpha,\infty}:=\inf\{t\in(\alpha,\infty),\ d[\alpha,t]=d[\alpha,\infty)\}. \tag{2.23}$$

Remark 2.7. We note that the subspace $\Lambda[\alpha,t]$, resp. $\Lambda[\alpha,\infty)$ is closely related with the matrix $S_\alpha(t)$ in (2.11). More precisely, under (1.1) for every conjoined basis (X,U) of (H) with constant kernel on $[\alpha,\infty)$ the corresponding matrices $P_{\mathcal{S}_\alpha}(t)$ and $P_{\mathcal{S}_\alpha\infty}$ in (2.12) satisfy

$$\left.\begin{aligned}\operatorname{Im}X(\alpha)\,P_{\mathcal{S}_\alpha}(t)=\big(\Lambda_\alpha[\alpha,t]\big)^\perp,\quad t\in(\alpha,\infty),\\ \operatorname{Im}X(\alpha)\,P_{\mathcal{S}_\alpha\infty}=\big(\Lambda_\alpha[\alpha,\infty)\big)^\perp.\end{aligned}\right\} \tag{2.24}$$

The proof of the first formula in (2.24) is based on (2.16) and [28, Equation 5.6] in which we showed that $\Lambda_\alpha[\alpha,t]=\operatorname{Ker}\hat{X}_\alpha(t)$ holds on $[\alpha,\infty)$, where $(\hat{X}_\alpha,\hat{U}_\alpha)$ is the principal solution at the point α. The second identity in (2.24) follows from the first one by using (2.12) and (2.22). Moreover, in [28, Theorem 5.2 and Remark 5.3] we proved the equalities

$$\operatorname{Im}S_\alpha(t)=\operatorname{Im}P_{\mathcal{S}_\alpha\infty},\quad \operatorname{rank}S_\alpha(t)=\operatorname{rank}P_{\mathcal{S}_\alpha\infty}=n-d[\alpha,\infty) \tag{2.25}$$

on $(\tau_{\alpha,\infty},\infty)$ with $\tau_{\alpha,\infty}$ defined in (2.23).

Throughout this paper we will consider only the intervals $[\alpha,\infty)$ with the maximal order of abnormality d_∞ defined in (2.18). The next remark shows how this condition reflects the properties of S-matrices corresponding to conjoined bases of (H) with constant kernel.

Remark 2.8. (i) Assume (1.1) and let (X,U) be a conjoined basis of (H) with constant kernel on $[\alpha,\infty)$. In [26, Theorem 4.1.12] we proved that the condition $d[\alpha,\infty)=d_\infty$ holds if and only if the matrix $S_\alpha(t)$ in (2.11) associated with (X,U) satisfies the equalities

$$\operatorname{Im}[\,P_{\mathcal{S}_{\alpha\infty}}-S_\alpha(t)\,T_\alpha]=\operatorname{Im}P_{\mathcal{S}_{\alpha\infty}}=\operatorname{Im}[\,P_{\mathcal{S}_{\alpha\infty}}-S_\alpha(t)\,T_\alpha]^T \tag{2.26}$$

for every $t\in[\alpha,\infty)$, where $P_{\mathcal{S}_{\alpha\infty}}$ and T_α are corresponding matrices in (2.12) and (2.13). We note that the identities in (2.26) can be equivalently replaced by

$$\operatorname{rank}[\,P_{\mathcal{S}_{\alpha\infty}}-S_\alpha(t)\,T_\alpha]=n-d[\alpha,\infty)\quad\text{on }[\alpha,\infty),$$

see [28, Theorem 6.9]. In addition, by [28, Equation 5.13] the conjoined basis (X,U) satisfies the conditions

$$n-d_\infty=n-d[\alpha,\infty)\le\operatorname{rank}X(t)\le n\quad\text{for all }t\in[\alpha,\infty). \tag{2.27}$$

(ii) Let T_β be the T-matrix in (2.13), which is associated with (X,U) on the interval $[\beta,\infty)$ for $\beta\in[\alpha,\infty)$. In [26, Theorem 4.3.1(ii)] we showed that the set $\operatorname{Im}T_\beta$ is constant in β on $[\alpha,\infty)$ if and only if the condition $d[\alpha,\infty)=d_\infty$ holds.

The following theorem is an extension of the result presented in Remark 2.8(i).

Theorem 2.9. *Assume* (1.1) *and let* (X,U) *be a conjoined basis of* (H) *with constant kernel on* $[\alpha,\infty)$. *Moreover, let* P, $S_\alpha(t)$, *and* T_α *be its corresponding matrices in* (2.5), (2.11), *and* (2.13), *respectively. Then the condition* $d[\alpha,\infty)=d_\infty$ *is equivalent with the formulas*

$$\operatorname{Im}[\,P-S_\alpha(t)\,T_\alpha]=\operatorname{Im}P=\operatorname{Im}[\,P-S_\alpha(t)\,T_\alpha]^T,\quad t\in[\alpha,\infty). \tag{2.28}$$

Proof. First we remark that by $P\,T_\alpha=T_\alpha$ and $PS_\alpha(t)=S_\alpha(t)$ on $[\alpha,\infty)$ we always have the inclusions

$$\operatorname{Im}[\,P-S_\alpha(t)\,T_\alpha]\subseteq\operatorname{Im}P,\quad \operatorname{Im}[\,P-S_\alpha(t)\,T_\alpha]^T=\operatorname{Im}[\,P-T_\alpha\,S_\alpha(t)]\subseteq\operatorname{Im}P \tag{2.29}$$

for every $t\in[\alpha,\infty)$. And since $\operatorname{rank}[P-S_\alpha(t)\,T_\alpha]=\operatorname{rank}[P-S_\alpha(t)\,T_\alpha]^T$ for all $t\in[\alpha,\infty)$, it is sufficient to show the equivalence of $d[\alpha,\infty)=d_\infty$ and the equality $\operatorname{Im}[P-S_\alpha(t)\,T_\alpha]=\operatorname{Im}P$ on $[\alpha,\infty)$. Assume that $d[\alpha,\infty)=d_\infty$ holds. Fix $t\in[\alpha,\infty)$ and let $v\in\operatorname{Ker}[P-S_\alpha(t)\,T_\alpha]^T=\operatorname{Ker}[P-T_\alpha\,S_\alpha(t)]$, that is, $[P-T_\alpha\,S_\alpha(t)]\,v=0$. Using the latter equation and the identities $P_{\mathcal{S}_{\alpha\infty}}\,P=P_{\mathcal{S}_{\alpha\infty}}$ and $P_{\mathcal{S}_{\alpha\infty}}\,T_\alpha=T_\alpha$ yields $[P_{\mathcal{S}_{\alpha\infty}}-T_\alpha\,S_\alpha(t)]\,v=P_{\mathcal{S}_{\alpha\infty}}\,[P-T_\alpha\,S_\alpha(t)]\,v=0$ and hence, the vector $v\in\operatorname{Ker}[P_{\mathcal{S}_{\alpha\infty}}-T_\alpha\,S_\alpha(t)]=\operatorname{Ker}[P_{\mathcal{S}_{\alpha\infty}}-S_\alpha(t)\,T_\alpha]^T=\operatorname{Ker}P_{\mathcal{S}_{\alpha\infty}}$, by the first equality in (2.26). Moreover, with the aid of identity $S_\alpha(t)=S_\alpha(t)\,P_{\mathcal{S}_{\alpha\infty}}$ we then get

$$Pv=[P-T_\alpha\,S_\alpha(t)]\,v+T_\alpha\,S_\alpha(t)\,v=[P-T_\alpha\,S_\alpha(t)]\,v+T_\alpha\,S_\alpha(t)\,P_{\mathcal{S}_{\alpha\infty}}\,v=0,$$

which shows that $v\in\operatorname{Ker}P$. Therefore, the inclusion $\operatorname{Ker}[P-S_\alpha(t)\,T_\alpha]^T\subseteq\operatorname{Ker}P$, or equivalently, the inclusion $\operatorname{Im}P\subseteq\operatorname{Im}[P-S_\alpha(t)\,T_\alpha]$ holds for every $t\in[\alpha,\infty)$. Combining the latter relation with the first property in (2.29) gives the equality $\operatorname{Im}[P-S_\alpha(t)\,T_\alpha]=\operatorname{Im}P$ on $[\alpha,\infty)$. Conversely, if $\operatorname{Im}[P-S_\alpha(t)\,T_\alpha]=\operatorname{Im}P$ is satisfied for all $t\in[\alpha,\infty)$, then

$$\operatorname{Im}[\,P_{\mathcal{S}_{\alpha\infty}}-S_\alpha(t)\,T_\alpha]=\operatorname{Im}P_{\mathcal{S}_{\alpha\infty}}\,[\,P-S_\alpha(t)\,T_\alpha]=\operatorname{Im}P_{\mathcal{S}_{\alpha\infty}}\,P=\operatorname{Im}P_{\mathcal{S}_{\alpha\infty}}$$

on $[\alpha,\infty)$, showing the first identity in (2.26). Finally, the condition $d[\alpha,\infty)=d_\infty$ holds by Remark 2.8(i), which completes the proof. □

The next statement is a combination of [28, Theorem 4.6] and [26, Theorem 2.3.3]. We again note that the Legendre condition (1.1) is in this case not needed.

Proposition 2.10. *Let (X,U) be a conjoined basis of (H) with constant kernel on $[\alpha,\infty)$ and let P be its corresponding orthogonal projector in (2.5). Moreover, let $(\tilde X,\tilde U)$ be a solution of (H), which is expressed in terms of (X,U) via matrices $M,N\in\mathbb{R}^{n\times n}$, that is,*

$$\begin{pmatrix}\tilde X\\ \tilde U\end{pmatrix}=\begin{pmatrix}X & \bar X\\ U & \bar U\end{pmatrix}\begin{pmatrix}M\\ N\end{pmatrix}\qquad \text{on } [\alpha,\infty), \tag{2.30}$$

where $(\bar X,\bar U)$ is a conjoined basis of (H) satisfying (2.1) and (2.14) with regard to (X,U). Then the inclusion $\operatorname{Im}\tilde X(\alpha)\subseteq\operatorname{Im}X(\alpha)$ holds if and only if $\operatorname{Im}N\subseteq\operatorname{Im}P$. In this case the matrices M and N do not depend on the particular choice of $(\bar X,\bar U)$. In addition, if $(\tilde X,\tilde U)$ is a conjoined basis with constant kernel on $[\alpha,\infty)$ and the equality $\operatorname{Im}\tilde X(\alpha)=\operatorname{Im}X(\alpha)$ holds, then

$$M \text{ is nonsingular},\quad M^TN=N^TM,\quad \operatorname{Im}N^T\subseteq\operatorname{Im}\tilde P, \tag{2.31}$$

where $\tilde P$ is the matrix in (2.5) associated with $(\tilde X,\tilde U)$.

Remark 2.11. (i) Combining (2.14) with formulas (2.15) and (2.30) at $t=\alpha$ we obtain that the solutions (X,U) and $(\tilde X,\tilde U)$ in Proposition 2.10 satisfy, see also [26, Equation (2.52)],

$$\tilde X(\alpha)=X(\alpha)\,M,\quad \tilde U(\alpha)=U(\alpha)\,M+X^{\dagger T}(\alpha)\,N,\quad \operatorname{Im}N\subseteq\operatorname{Im}P.$$

The first equality in (2.15) allows to rewrite the expression for the matrix $\tilde X(t)$ in (2.30) into the form

$$\tilde X(t)=X(t)\,[M+S_\alpha(t)\,N]=X(t)\,[PM+S_\alpha(t)\,N]\quad \text{on } [\alpha,\infty), \tag{2.32}$$

where $S_\alpha(t)$ is the S-matrix in (2.11) associated with (X,U). In particular, this shows that the inclusion $\operatorname{Im}\tilde X(t)\subseteq\operatorname{Im}X(t)$ holds for every $t\in[\alpha,\infty)$, see [26, Theorem 2.3.3]. We also note that the matrix N is the (constant) Wronskian of (X,U) and $(\tilde X,\tilde U)$.

(ii) Let $(\tilde X,\tilde U)$ be a conjoined basis of (H) with constant kernel on $[\alpha,\infty)$ such that $\operatorname{Im}\tilde X(\alpha)=\operatorname{Im}X(\alpha)$ holds. Then $\operatorname{Im}\tilde X(t)=\operatorname{Im}X(t)$ on $[\alpha,\infty)$, as we proved in [28, Theorem 4.10]. If $\tilde S_\alpha(t)$ is the S-matrix which corresponds to $(\tilde X,\tilde U)$ on $[\alpha,\infty)$, then we have identities

$$[PM+S_\alpha(t)\,N]^\dagger=\tilde PM^{-1}-\tilde S_\alpha(t)\,N^T, \tag{2.33}$$

$$\operatorname{Im}[PM+S_\alpha(t)\,N]=\operatorname{Im}P,\quad \operatorname{Im}[PM+S_\alpha(t)\,N]^T=\operatorname{Im}\tilde P, \tag{2.34}$$

$$\tilde S_\alpha(t)=[PM+S_\alpha(t)\,N]^\dagger\,S_\alpha(t)\,M^{T-1}\tilde P,\quad \operatorname{Im}\tilde S_\alpha(t)=\operatorname{Im}\tilde PM^{-1}S_\alpha(t) \tag{2.35}$$

for every $t\in[\alpha,\infty)$, see [28, Remark 4.7, Theorem 4.10]. In particular, since $S_\alpha(\alpha)=0=\tilde S_\alpha(\alpha)$ by (2.11), formula (2.33) at $t=\alpha$ and the inclusion in (2.31) give the equalities

$$(PM)^\dagger=\tilde PM^{-1},\quad N(PM)^\dagger=N\tilde PM^{-1}=NM^{-1}. \tag{2.36}$$

Moreover, the identities in (2.32) and (2.34) yield $X(t) = \tilde{X}(t)\,[PM + S_\alpha(t)\,N]^\dagger$ and the formulas for the pseudoinverses

$$\left.\begin{aligned} \tilde{X}^\dagger(t) &= [PM + S_\alpha(t)\,N]^\dagger\, X^\dagger(t),\\ X^\dagger(t) &= [PM + S_\alpha(t)\,N]\,\tilde{X}^\dagger(t), \end{aligned}\right\} \quad \text{on } [\alpha,\infty). \tag{2.37}$$

3. **Principal solutions at infinity.** Following [29, Definition 7.1], we say that a conjoined basis $(\hat{X},\hat{U})$ of (H) is a *principal solution at infinity* if $(\hat{X},\hat{U})$ has constant kernel on $[\alpha,\infty)$ and its corresponding matrix $\hat{S}_\alpha(t)$ defined in (2.11) through $\hat{X}(t)$ satisfies $\hat{S}_\alpha^\dagger(t) \to 0$ as $t\to\infty$, that is, $\hat{T}_\alpha = 0$ in (2.13). In this case we will say that $(\hat{X},\hat{U})$ is a principal solution of (H) at infinity with respect to the interval $[\alpha,\infty)$. By (2.27), the principal solutions of (H) can be classified according to the rank of $\hat{X}(t)$ on $[\alpha,\infty)$. In particular, the *minimal principal solution* $(\hat{X}_{\min},\hat{U}_{\min})$ of (H) at infinity satisfies $\operatorname{rank}\hat{X}_{\min}(t) = n - d_\infty$, while the *maximal principal solution* $(\hat{X}_{\max},\hat{U}_{\max})$ of (H) at infinity is determined by $\operatorname{rank}\hat{X}_{\max}(t) = n$, hence $\hat{X}_{\max}(t)$ is invertible on $[\alpha,\infty)$, see [29, Remark 7.2].

In the next proposition we recall from [29, Theorem 7.6] and [28, Theorems 7.6] the characterization of the nonoscillation of system (H) by the existence of a principal solution of (H) at infinity with any possible rank, as well as the uniqueness of the minimal principal solution.

Proposition 3.1. *Assume that* (1.1) *holds. Then the following statements are equivalent.*

(i) *System* (H) *is nonoscillatory.*
(ii) *There exists a principal solution of* (H) *at infinity.*
(iii) *For any integer r satisfying $n-d_\infty \le r \le n$ there exists a principal solution of* (H) *at infinity with rank equal to r.*

In particular, system (H) *is nonoscillatory if and only if there exists a minimal principal solution of* (H) *at infinity. In this case the minimal principal solution is unique up to a right nonsingular constant multiple.*

In [29, Equation 7.4] we defined for a nonoscillatory system (H) the point $\hat{\alpha}_{\min} \in [a,\infty)$ by

$$\hat{\alpha}_{\min} := \inf\big\{\alpha\in[a,\infty),\ (\hat{X}_{\min},\hat{U}_{\min}) \text{ has constant kernel on } [\alpha,\infty)\big\}, \tag{3.1}$$

where $(\hat{X}_{\min},\hat{U}_{\min})$ is the minimal principal solution of (H) at infinity. We note that the equality $d[\alpha,\infty) = d_\infty$ holds for every $\alpha > \hat{\alpha}_{\min}$, see [29, Theorem 7.9]. In turn, combining this fact with formula (2.19) we obtain that

$$d[\hat{\alpha}_{\min},\infty) = d_\infty, \quad \text{i.e.,} \quad \hat{\alpha}_{\min} \ge \alpha_\infty.$$

The next results are based on [28, Theorem 7.5] and [29, Lemma 7.5 and Remark 7.11].

Proposition 3.2. *Assume that* (1.1) *holds and system* (H) *is nonoscillatory with $\hat{\alpha}_{\min}$ defined in* (3.1). *Then the following statements hold.*

(i) *If $(\hat{X},\hat{U})$ is a principal solution of* (H) *at infinity with respect to the interval $[\alpha,\infty)$, then $d[\alpha,\infty) = d_\infty$. Moreover, the pair $(\hat{X},\hat{U})$ is a principal solution of* (H) *at infinity also with respect to the interval $[\beta,\infty)$ for every $\beta\ge\alpha$.*

(ii) *Every principal solution* $(\hat{X}, \hat{U})$ *of* (H) *at infinity is a principal solution with respect to* $[\alpha, \infty)$ *for every* $\alpha \in (\hat{\alpha}_{\min}, \infty)$. *In other words, the conjoined basis* $(\hat{X}, \hat{U})$ *has constant kernel on the open interval* $(\hat{\alpha}_{\min}, \infty)$ *and its corresponding matrix* $\hat{S}_\alpha(t)$ *in* (2.11) *satisfies* $\hat{S}_\alpha^\dagger(t) \to 0$ *as* $t \to \infty$ *for every* $\alpha > \hat{\alpha}_{\min}$.

Remark 3.3. We note that the orthogonal projector $P_{\hat{\mathcal{S}}_\alpha \infty}$ in (2.12) associated with the principal solution $(\hat{X}, \hat{U})$ through the matrix $\hat{S}_\alpha(t)$ is the same for all initial points $\alpha \in (\hat{\alpha}_{\min}, \infty)$, see [29, Remark 7.11]. Therefore, we will use the notation

$$P_{\hat{\mathcal{S}}\infty} := P_{\hat{\mathcal{S}}_\alpha \infty} \quad \text{for } \alpha \in (\hat{\alpha}_{\min}, \infty). \tag{3.2}$$

Given a principal solution $(\hat{X}, \hat{U})$ of (H) at infinity, we define the point $\hat{\alpha} \in [a, \infty)$ associated with $(\hat{X}, \hat{U})$ by

$$\begin{aligned} \hat{\alpha} := \inf \big\{ \alpha \in [a, \infty),\ (\hat{X}, \hat{U}) \text{ is a principal solution} \\ \text{of (H) with respect to } [\alpha, \infty) \big\}. \end{aligned} \tag{3.3}$$

From Proposition 3.2 it immediately follows that the point $\hat{\alpha}$ in (3.3) satisfies the inequalities $\alpha_\infty \le \hat{\alpha} \le \hat{\alpha}_{\min}$ with α_∞ defined in (2.19). We also note that the set $(\hat{\alpha}, \infty)$ is the maximal open interval with the property that $(\hat{X}, \hat{U})$ is a principal solution of (H) with respect to $[\alpha, \infty)$ for every $\alpha \in (\hat{\alpha}, \infty)$. Therefore, we will often say that $(\hat{X}, \hat{U})$ is a principal solution of (H) at infinity with respect to the maximal interval $(\hat{\alpha}, \infty)$. In particular, the conjoined basis $(\hat{X}, \hat{U})$ has constant kernel on the open interval $(\hat{\alpha}, \infty)$ and the S-matrix $\hat{S}_\alpha(t)$ in (2.11) associated with $(\hat{X}, \hat{U})$ satisfies $\hat{S}_\alpha^\dagger(t) \to 0$ as $t \to \infty$ for every $\alpha > \hat{\alpha}$. In the next theorem we derive an exact relation between the points $\hat{\alpha}$ and $\hat{\alpha}_{\min}$.

Theorem 3.4. *Assume that* (1.1) *holds and system* (H) *is nonoscillatory with* $\hat{\alpha}_{\min}$ *defined in* (3.1). *Let* $(\hat{X}, \hat{U})$ *be a principal solution of* (H) *at infinity and let* $\hat{\alpha}$ *be its corresponding point in* (3.3). *Then the equality* $\hat{\alpha} = \hat{\alpha}_{\min}$ *holds.*

Proof. Let $(\hat{X}, \hat{U})$, $\hat{\alpha}$, and $\hat{\alpha}_{\min}$ be as in the proposition and suppose that $\hat{\alpha} < \hat{\alpha}_{\min}$. According to (3.3) there exists a point $\beta \in (\hat{\alpha}, \hat{\alpha}_{\min})$ such that $(\hat{X}, \hat{U})$ is a principal solution of (H) at infinity with respect to the interval $[\beta, \infty)$. By Proposition 3.2(i) with $\alpha := \beta$ we know that $d[\beta, \infty) = d_\infty$. Let $(\hat{X}_{\min}, \hat{U}_{\min})$ be the minimal principal solution of (H) at infinity. By [29, Theorem 7.3] it follows that the pair $(\hat{X}_{\min}, \hat{U}_{\min})$ is a minimal principal solution at infinity with respect to the interval $[\beta, \infty)$. For this we note that $(\hat{X}_{\min}, \hat{U}_{\min})$ is contained in $(\hat{X}, \hat{U})$ on $[\beta, \infty)$ according to the properties of the relation "being contained" in [28, Section 5]. The uniqueness of the minimal principal solution and the definition of $\hat{\alpha}_{\min}$ in (3.1) then yield that $\beta \ge \hat{\alpha}_{\min}$, which is a contradiction. Therefore, the equality $\hat{\alpha} = \hat{\alpha}_{\min}$ holds and the proof is complete. ☐

In the following result we present a construction of a principal solution of (H) at infinity from a conjoined basis of (H) with constant kernel on $[\alpha, \infty)$. It is a generalization of [32, Equation (10)], where only the minimal principal solution of (H) was considered. This result will be utilized for the construction of a distinguished solution of ($\mathcal{R}$) at infinity in Theorem 7.16.

Theorem 3.5. *Assume that condition* (1.1) *holds and system* (H) *is nonoscillatory. Let* $\alpha \in [a, \infty)$ *be such that* $d[\alpha, \infty) = d_\infty$ *and let there exists a conjoined basis of*

(H) *with constant kernel on* $[\alpha,\infty)$. *Then a solution* $(\hat{X},\hat{U})$ *of* (H) *is a principal at solution infinity with respect to the interval* $[\alpha,\infty)$ *if and only if*

$$\begin{pmatrix}\hat{X}\\ \hat{U}\end{pmatrix} := \begin{pmatrix}X & \bar{X}\\ U & \bar{U}\end{pmatrix}\begin{pmatrix}I\\ -T_\alpha\end{pmatrix} \qquad \text{on } [\alpha,\infty), \tag{3.4}$$

for some conjoined basis (X,U) *of* (H) *with constant kernel on* $[\alpha,\infty)$. *Here the conjoined basis* $(\bar{X},\bar{U})$ *and the matrix* T_α *are associated with* (X,U) *through Remark 2.6 and* (2.13).

Proof. If $(\hat{X},\hat{U})$ is a principal solution of (H) at infinity with respect to $[\alpha,\infty)$, then $(\hat{X},\hat{U})$ has constant kernel on $[\alpha,\infty)$ and the associated matrix $\hat{T}_\alpha$ in (2.13) satisfies $\hat{T}_\alpha=0$. Formula (3.4) then holds trivially with $(X,U):=(\hat{X},\hat{U})$. Conversely, let (X,U) be a conjoined basis of (H) with constant kernel on $[\alpha,\infty)$ and let P, $S_\alpha(t)$, $P_{\mathcal{S}_\alpha\infty}$, and T_α be the matrices in (2.5), (2.11), (2.12), and (2.13) corresponding to (X,U) on $[\alpha,\infty)$. Consider the matrix solution $(\hat{X},\hat{U})$ of (H) in (3.4). Since $P\,T_\alpha=T_\alpha$, it follows from Proposition 2.10 with $M:=I$, $N:=-T_\alpha$ and $(\tilde{X},\tilde{U}):=(\hat{X},\hat{U})$ that $(\hat{X},\hat{U})$ is a conjoined basis of (H), which in turn by (2.32) yields that $\hat{X}(t)=X(t)\,[P-S_\alpha(t)\,T_\alpha]$ on $[\alpha,\infty)$. Then, by $PS_\alpha(t)=S_\alpha(t)$ and using (2.28) from Theorem 2.9, we get

$$\begin{aligned}\operatorname{Ker}\hat{X}(t) &= \operatorname{Ker}\,X^\dagger(t)\,X(t)\,[P-S_\alpha(t)\,T_\alpha] = \operatorname{Ker}\,P\,[P-S_\alpha(t)\,T_\alpha]\\ &= \operatorname{Ker}\,[P-S_\alpha(t)\,T_\alpha] \overset{(2.28)}{=} (\operatorname{Im}P)^\perp = \operatorname{Ker}P\end{aligned}$$

on $[\alpha,\infty)$. This shows that $(\hat{X},\hat{U})$ has constant kernel on $[\alpha,\infty)$ as well. Moreover, if $\hat{P}$, $\hat{S}_\alpha(t)$ and $P_{\hat{\mathcal{S}}_\alpha\infty}$ are the matrices in (2.5), (2.11), and (2.12) corresponding to $(\hat{X},\hat{U})$ on $[\alpha,\infty)$, then by using the first equations in (2.36) and (2.35), respectively, we have that $\hat{P}=P$ and

$$\hat{S}_\alpha(t) = [P-S_\alpha(t)\,T_\alpha]^\dagger\,S_\alpha(t)\,P = [P-S_\alpha(t)\,T_\alpha]^\dagger\,S_\alpha(t) \tag{3.5}$$

for all $t\in[\alpha,\infty)$. On the other hand, applying the second identity in (2.35) yields the equalities $\operatorname{Im}\hat{S}_\alpha(t)=\operatorname{Im}\hat{P}S_\alpha(t)=\operatorname{Im}PS_\alpha(t)=\operatorname{Im}S_\alpha(t)$ on $[\alpha,\infty)$, which in particular by (2.12) imply that $P_{\hat{\mathcal{S}}_\alpha\infty}=P_{\mathcal{S}_\alpha\infty}$. Let $\tau_{\alpha,\infty}$ be defined in (2.23). Then $S_\alpha(t)\,S_\alpha^\dagger(t)=P_{\mathcal{S}_\alpha\infty}=S_\alpha^\dagger(t)\,S_\alpha(t)$ and $\hat{S}_\alpha(t)\,\hat{S}_\alpha^\dagger(t)=P_{\hat{\mathcal{S}}_\alpha\infty}=\hat{S}_\alpha^\dagger(t)\,\hat{S}_\alpha(t)$ on $(\tau_{\alpha,\infty},\infty)$, by (2.25). Consequently, with the aid of (2.28) and (3.5) together with the identity $T_\alpha\,P_{\mathcal{S}_\alpha\infty}=T_\alpha=P_{\mathcal{S}_\alpha\infty}\,T_\alpha$ we obtain that

$$\begin{aligned}\hat{S}_\alpha^\dagger(t) &= P_{\mathcal{S}_\alpha\infty}\,\hat{S}_\alpha^\dagger(t) = S_\alpha^\dagger(t)\,S_\alpha(t)\,\hat{S}_\alpha^\dagger(t) = S_\alpha^\dagger(t)\,PS_\alpha(t)\,\hat{S}_\alpha^\dagger(t)\\ &\overset{(2.28)}{=} S_\alpha^\dagger(t)\,[P-S_\alpha(t)\,T_\alpha]\,[P-S_\alpha(t)\,T_\alpha]^\dagger\,S_\alpha(t)\,\hat{S}_\alpha^\dagger(t)\\ &\overset{(3.5)}{=} S_\alpha^\dagger(t)\,[P-S_\alpha(t)\,T_\alpha]\,\hat{S}_\alpha(t)\,\hat{S}_\alpha^\dagger(t) = S_\alpha^\dagger(t)\,[P-S_\alpha(t)\,T_\alpha]\,P_{\hat{\mathcal{S}}_\alpha\infty}\\ &= S_\alpha^\dagger(t)\,[P-S_\alpha(t)\,T_\alpha]\,P_{\mathcal{S}_\alpha\infty} = S_\alpha^\dagger(t)-T_\alpha\end{aligned} \tag{3.6}$$

for every $t\in(\tau_{\alpha,\infty},\infty)$. Finally, formula (3.6) implies that $\hat{S}_\alpha^\dagger(t)\to 0$ as $t\to\infty$. This shows that the conjoined basis $(\hat{X},\hat{U})$ is a principal solution of (H) at infinity with respect to the interval $[\alpha,\infty)$. □

Remark 3.6. It follows from Proposition 2.10 and Remark 2.11(i) that the principal solution $(\hat{X},\hat{U})$ constructed in (3.4) satisfies the equality $\operatorname{Im}\hat{X}(\alpha)=\operatorname{Im}X(\alpha)$.

Moreover, as we noted in Remark 2.11(ii), this condition is valid on the whole interval $[\alpha,\infty)$, i.e., $\operatorname{Im}\hat{X}(t)=\operatorname{Im}X(t)$ holds for all $t\in[\alpha,\infty)$. In particular, the last equality means that the conjoined bases (X,U) and $(\hat{X},\hat{U})$ belong to the same genus of conjoined bases of (H) as we define below, see also Remark 3.13.

In the second part of this section we recall basic concepts from the theory of genera of conjoined bases of (H) from our recent work [27, Section 4]. We wish to point out that in this context the Legendre condition (1.1) is not assumed and/or system (H) is allowed to be oscillatory. Define the orthogonal projector

$$R_{\Lambda\infty}(t):=\mathcal{P}_{\mathcal{W}_t^{\perp}},\quad \text{where}\quad \mathcal{W}_t:=\Lambda_t[\alpha_\infty,\infty),\quad t\in[\alpha_\infty,\infty),\tag{3.7}$$

where the point α_∞ is determined in (2.19) and the subspace $\Lambda_t[\alpha_\infty,\infty)$ is defined in (2.17). From the second identity in (2.20) it follows that for any $\alpha\geq\alpha_\infty$ the matrix $R_{\Lambda\infty}(t)$ defined in (3.7) is the orthogonal projector onto the set $\big(\Lambda_t[\alpha,\infty)\big)^{\perp}$ on $[\alpha,\infty)$, i.e.,

$$R_{\Lambda\infty}(t)=\mathcal{P}_{\mathcal{U}_t^{\perp}},\quad \text{where}\quad \mathcal{U}_t:=\Lambda_t[\alpha,\infty),\quad t\in[\alpha,\infty).\tag{3.8}$$

Remark 3.7. Assume the Legendre condition (1.1). Let (X,U) be a conjoined basis of (H) with constant kernel on $[\alpha,\infty)\subseteq[\alpha_\infty,\infty)$ and let P and $P_{\mathcal{S}_\alpha\infty}$ be the corresponding orthogonal projectors in (2.4) and (2.12), respectively. Combining (2.24) and (3.8) then yields the identity

$$\operatorname{Im}X(\alpha)\,P_{\mathcal{S}_\alpha\infty}=\operatorname{Im}R_{\Lambda\infty}(\alpha).\tag{3.9}$$

Moreover, since $\operatorname{Im}[X(\alpha)\,P_{\mathcal{S}_\alpha\infty}]^T=P_{\mathcal{S}_\alpha\infty}$, $PP_{\mathcal{S}_\alpha\infty}=P_{\mathcal{S}_\alpha\infty}$, and $X^\dagger(\alpha)\,X(\alpha)=P$, we have that

$$\begin{aligned}[X(\alpha)\,P_{\mathcal{S}_\alpha\infty}]^\dagger&=PP_{\mathcal{S}_\alpha\infty}\,[X(\alpha)\,P_{\mathcal{S}_\alpha\infty}]^\dagger=X^\dagger(\alpha)\,X(\alpha)\,P_{\mathcal{S}_\alpha\infty}\,[X(\alpha)\,P_{\mathcal{S}_\alpha\infty}]^\dagger\\&=X^\dagger(\alpha)\,[X(\alpha)\,P_{\mathcal{S}_\alpha\infty}]\,[X(\alpha)\,P_{\mathcal{S}_\alpha\infty}]^\dagger\overset{(3.9)}{=}X^\dagger(\alpha)\,R_{\Lambda\infty}(\alpha).\end{aligned}\tag{3.10}$$

The orthogonal projector $R_{\Lambda\infty}(t)$ defined in (3.7) plays a crucial role in the following notion. According to [27, Definition 4.3] we say that two conjoined bases (X_1,U_1) and (X_2,U_2) of (H) *have the same genus* (or they *belong to the same genus*) if there exists $\alpha\in[\alpha_\infty,\infty)$ such that

$$\operatorname{Im}X_1(t)+\operatorname{Im}R_{\Lambda\infty}(t)=\operatorname{Im}X_2(t)+\operatorname{Im}R_{\Lambda\infty}(t),\quad t\in[\alpha,\infty).$$

From this definition it follows that the relation “having (or belonging to) the same genus” is an equivalence on the set of all conjoined bases of (H). Therefore, there exists a partition of this set into disjoint classes of conjoined bases of (H) with the same genus. This allows to interpret each such an equivalence class $\mathcal{G}$ as a genus itself. The following result is proven in [27, Theorem 4.5].

Proposition 3.8. *Let (X_1,U_1) and (X_2,U_2) be conjoined bases of (H). Then the following statements are equivalent.*

(i) *The conjoined bases (X_1,U_1) and (X_2,U_2) belong to the same genus.*
(ii) *The equality $\operatorname{Im}X_1(t)+\operatorname{Im}R_{\Lambda\infty}(t)=\operatorname{Im}X_2(t)+\operatorname{Im}R_{\Lambda\infty}(t)$ is satisfied for every $t\in[\alpha_\infty,\infty)$.*
(iii) *The equality $\operatorname{Im}X_1(t)+\operatorname{Im}R_{\Lambda\infty}(t)=\operatorname{Im}X_2(t)+\operatorname{Im}R_{\Lambda\infty}(t)$ is satisfied for some $t\in[\alpha_\infty,\infty)$.*

Let $\mathcal{G}$ be a genus of conjoined bases of (H) and let (X,U) be a conjoined basis belonging to $\mathcal{G}$. The results in Proposition 3.8 imply that for all $t\in[\alpha_\infty,\infty)$ the subspace $\operatorname{Im} X(t)+\operatorname{Im} R_{\Lambda\infty}(t)$ does not depend on the particular choice of (X,U) in $\mathcal{G}$. Therefore, the orthogonal projector onto $\operatorname{Im} X(t)+\operatorname{Im} R_{\Lambda\infty}(t)$, i.e., the matrix

$$R_{\mathcal{G}}(t):=\mathcal{P}_{\mathcal{V}_t}, \quad \text{where} \quad \mathcal{V}_t:=\operatorname{Im} X(t)+\operatorname{Im} R_{\Lambda\infty}(t), \quad t\in[\alpha_\infty,\infty), \tag{3.11}$$

is uniquely determined for each genus $\mathcal{G}$. The next statement is from [27, Theorem 4.7].

Proposition 3.9. *Let $\mathcal{G}$ be a genus of conjoined bases of* (H) *and let $R_{\mathcal{G}}(t)$ be the orthogonal projector defined in* (3.11)*. Then the matrix $R_{\mathcal{G}}(t)$ is a solution of the Riccati equation* (1.2) *on $[\alpha_\infty,\infty)$ and the inclusion $\operatorname{Im} R_{\Lambda\infty}(t)\subseteq \operatorname{Im} R_{\mathcal{G}}(t)$ holds for every $t\in[\alpha_\infty,\infty)$.*

Remark 3.10. If the orthogonal projector $R_{\mathcal{G}}(t)$ satisfies $R_{\mathcal{G}}(t)=R_{\Lambda\infty}(t)$ on $[\alpha_\infty,\infty)$, then the genus $\mathcal{G}=\mathcal{G}_{\min}$ is called *minimal*, while if $R_{\mathcal{G}}(t)\equiv I$ on $[\alpha_\infty,\infty)$, then the genus $\mathcal{G}=\mathcal{G}_{\max}$ is called *maximal*.

The next result describes important properties of nonoscillatory conjoined bases from a given genus $\mathcal{G}$. These properties will be utilized in Section 4 to show their connection with symmetric solutions of the Riccati equation ($\mathcal{R}$) associated with the genus $\mathcal{G}$, see Theorem 4.18.

Proposition 3.11. *Let $\mathcal{G}$ be a genus of conjoined basis of* (H) *with the corresponding orthogonal projector $R_{\mathcal{G}}(t)$ in* (3.11)*. Furthermore, let (X,U) be a conjoined basis of* (H) *with constant kernel on $[\alpha,\infty)\subseteq[\alpha_\infty,\infty)$ such that (X,U) belongs to $\mathcal{G}$ and let $R(t)$ and $Q(t)$ be the matrices in* (2.4) *and* (2.7)*. Then the equality $R_{\mathcal{G}}(t)=R(t)$ holds for all $t\in[\alpha,\infty)$. Moreover, the matrices $X(t)$, $X^\dagger(t)$, and $U(t)$ satisfy on $[\alpha,\infty)$ the equations*

$$X'=(\mathcal{A}+\mathcal{B}\,Q)\,X, \quad (X^\dagger)'=-X^\dagger(\mathcal{A}+\mathcal{B}\,Q), \tag{3.12}$$

$$U'=\mathcal{A}\,U+[C-(A+A^T)\,Q]\,X, \tag{3.13}$$

where the matrices $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are defined in (1.3)*.*

Proof. For the proof of $R_{\mathcal{G}}(t)=R(t)$ we refer to [27, Proposition 4.16]. We will prove that (3.12) and (3.13) hold. From the definition of the matrix $\mathcal{A}(t)$ in (1.3) it follows that $\mathcal{A}(t)\,R_{\mathcal{G}}(t)=A(t)\,R_{\mathcal{G}}(t)$ on $[\alpha,\infty)$. Moreover, using (1.3), (2.8), (2.9), and the identity $R_{\mathcal{G}}(t)\,X(t)=X(t)$ on $[\alpha,\infty)$ yields the formula

$$X' \overset{(2.9)}{=} AR_{\mathcal{G}}X+BRU \overset{(2.8)}{=} \mathcal{A}R_{\mathcal{G}}X+BQX \overset{(1.3)}{=} (\mathcal{A}+\mathcal{B}\,Q)X$$

on $[\alpha,\infty)$. Since the function $X^\dagger\in \mathrm{C}^1_{\mathrm{p}}$, equation (2.10) in Proposition 2.3 becomes

$$(X^\dagger)'=-X^\dagger[AR_{\mathcal{G}}-A^T(I-R_{\mathcal{G}})]-X^\dagger BQ \overset{(1.3)}{=} -X^\dagger(\mathcal{A}+\mathcal{B}\,Q)$$

on $[\alpha,\infty)$. Finally, by using $R_{\mathcal{G}}(t)\,U(t)=Q(t)\,X(t)$ for every $t\in[\alpha,\infty)$ we get

$$\begin{aligned} U'-\mathcal{A}\,U &\overset{(1.3)}{=} CX-A^TU-[AR_{\mathcal{G}}-A^T(I-R_{\mathcal{G}})]\,U=CX-(A+A^T)\,R_{\mathcal{G}}U \\ &\overset{(2.8)}{=} CX-(A+A^T)\,QX=[C-(A+A^T)\,Q]\,X \end{aligned}$$

on $[\alpha,\infty)$. Thus, the matrix $U(t)$ solves (3.13) on $[\alpha,\infty)$. The proof is complete. $\square$

Remark 3.12. Let $\Phi_\alpha(t)$ be the fundamental matrix of the system $Y'=[\mathcal{A}(t)+\mathcal{B}(t)\,Q(t)]\,Y$ for $t\in[\alpha,\infty)$ satisfying $\Phi_\alpha(\alpha)=I$. It is well-known that $\Phi_\alpha^{T-1}(t)$ is the fundamental matrix of the adjoint system $Y'=-[\mathcal{A}(t)+\mathcal{B}(t)\,Q(t)]^T\,Y$ for $t\in[\alpha,\infty)$. From (3.12) we then obtain by the uniqueness of solutions that

$$X(t)=\Phi_\alpha(t)\,X(\alpha), \quad X^\dagger(t)=X^\dagger(\alpha)\,\Phi_\alpha^{-1}(t), \quad t\in[\alpha,\infty). \tag{3.14}$$

Remark 3.13. In [29, Theorem 7.12] we proved that every genus $\mathcal{G}$ of conjoined bases of nonoscillatory system (H) contains a principal solution of (H) at infinity. Moreover, in Theorem 3.5 we described the construction of any such a principal solution in terms of conjoined bases from the genus $\mathcal{G}$, see also Remark 3.6.

In the next proposition we recall from [29, Theorem 7.13] a complete classification of all principal solutions of (H) at infinity within the genus $\mathcal{G}$.

Proposition 3.14. *Assume that* (1.1) *holds and system* (H) *is nonoscillatory with* $\hat{\alpha}_{\min}$ *defined in* (3.1). *Let* $(\hat{X}, \hat{U})$ *be a principal solution of* (H) *at infinity, which belongs to a genus* $\mathcal{G}$. *Moreover, let* $\hat{P}$ *and* $P_{\hat{\mathcal{S}}\infty}$ *be the orthogonal projectors defined through the function* $\hat{X}(t)$ *on* $(\hat{\alpha}_{\min}, \infty)$ *in* (2.5), (2.12), *and* 3.2. *Then a solution* (X, U) *of* (H) *is a principal solution belonging to* $\mathcal{G}$ *if and only if for some (and hence for every)* $\alpha \in (\hat{\alpha}_{\min}, \infty)$ *there exist matrices* $\hat{M}, \hat{N} \in \mathbb{R}^{n\times n}$ *such that*

$$X(\alpha) = \hat{X}(\alpha)\,\hat{M}, \quad U(\alpha) = \hat{U}(\alpha)\,\hat{M} + \hat{X}^{\dagger T}(\alpha)\,\hat{N},$$

$$\hat{M} \text{ is nonsingular}, \quad \hat{M}^T\hat{N} = \hat{N}^T\hat{M}, \quad \operatorname{Im}\hat{N} \subseteq \operatorname{Im}\hat{P}, \quad P_{\hat{\mathcal{S}}\infty}\,\hat{N}\hat{M}^{-1}P_{\hat{\mathcal{S}}\infty} = 0.$$

4. **Riccati matrix differential equation for given genus.**

In this section we present a new theory extending the results by Reid in [22, 23] about Riccati matrix differential equation (R) to general possibly uncontrollable systems (H). Namely, for every genus $\mathcal{G}$ of conjoined bases of (H) we consider the Riccati matrix differential equation ($\mathcal{R}$). In Lemma 4.1, Theorem 4.3, and Corollary 4.5 we first derive properties of solutions of ($\mathcal{R}$) in the relation with the associated projector $R_{\mathcal{G}}(t)$ in (3.11). In (4.9) and (4.18) we introduce an auxiliary linear differential system and the so-called F-matrix for a solution of this system, which serve as main tools for the formulation of the results in this section. In particular, in Theorem 4.16 we present additional properties of solutions of ($\mathcal{R}$) obtained through the above mentioned F-matrix. The main results concerning the correspondence between the solutions of the Riccati equation ($\mathcal{R}$) and conjoined bases of (H) from the genus $\mathcal{G}$ are contained in Theorems 4.18 and 4.21.

First we derive some auxiliary properties of the projector $R_{\mathcal{G}}(t)$, being a solution of the Riccati equation (1.2), and the coefficient $\mathcal{A}(t)$ in (1.3). In particular, we represent $R_{\mathcal{G}}(t)$ as a solution of a linear differential system

$$R_{\mathcal{G}}' = [\mathcal{A}(t),\, R_{\mathcal{G}}] = \mathcal{A}(t)\, R_{\mathcal{G}} - R_{\mathcal{G}}\, \mathcal{A}(t). \tag{4.1}$$

We note that since $R_{\mathcal{G}}(t)$ is symmetric, then it solves also the system

$$R_{\mathcal{G}}' = [R_{\mathcal{G}},\, \mathcal{A}^T(t)] = R_{\mathcal{G}}\, \mathcal{A}^T(t) - \mathcal{A}^T(t)\, R_{\mathcal{G}}. \tag{4.2}$$

being the transpose of the system in (4.1). For $M, N \in \mathbb{R}^{n\times n}$ the notation $[M, N]$ used in (4.1) and (4.2) means their commutator, i.e., $[M, N] := MN - NM$.

Lemma 4.1. *Let* $\mathcal{G}$ *be a genus of conjoined bases of* (H) *and let* $R_{\mathcal{G}}(t)$ *and* $\mathcal{A}(t)$ *be the corresponding matrices in* (3.11) *and* (1.3). *Then* $[R_{\mathcal{G}}(t),\, \mathcal{A}(t) + \mathcal{A}^T(t)] = 0$ *for every* $t \in [\alpha_\infty, \infty)$, *i.e., the matrices* $R_{\mathcal{G}}(t)$ *and* $\mathcal{A}(t) + \mathcal{A}^T(t)$ *commute on* $[\alpha_\infty, \infty)$. *Moreover, the orthogonal projector* $R_{\mathcal{G}}(t)$ *satisfies on* $[\alpha_\infty, \infty)$ *system* (4.1).

Proof. First we note that from the definition of the matrix $\mathcal{A}(t)$ in (1.3) we have on $[\alpha_\infty, \infty)$ the formulas

$$\mathcal{A}R_{\mathcal{G}} = AR_{\mathcal{G}}, \quad R_{\mathcal{G}}\mathcal{A}^T = R_{\mathcal{G}}A^T, \tag{4.3}$$

$$R_{\mathcal{G}}\mathcal{A} = R_{\mathcal{G}}AR_{\mathcal{G}} - R_{\mathcal{G}}A^T(I - R_{\mathcal{G}}) = R_{\mathcal{G}}\,(A + A^T)\,R_{\mathcal{G}} - R_{\mathcal{G}}A^T. \tag{4.4}$$

By combining equality (4.4) with the second identity in (4.3) we obtain that

$$\begin{aligned} R_{\mathcal{G}}\,(\mathcal{A}+\mathcal{A}^T) &= R_{\mathcal{G}}\mathcal{A} + R_{\mathcal{G}}\mathcal{A}^T = [R_{\mathcal{G}}\,(A+A^T)\,R_{\mathcal{G}} - R_{\mathcal{G}}A^T] + R_{\mathcal{G}}A^T \\ &= R_{\mathcal{G}}\,(A+A^T)\,R_{\mathcal{G}} \end{aligned} \tag{4.5}$$

on $[\alpha_\infty,\infty)$. From (4.5) it then follows that the matrix $R_{\mathcal{G}}(t)\,[\mathcal{A}(t)+\mathcal{A}^T(t)]$ is symmetric for every $t\in[\alpha_\infty,\infty)$, which in turn implies the equality

$$R_{\mathcal{G}}(t)\,[\mathcal{A}(t)+\mathcal{A}^T(t)] = [\mathcal{A}(t)+\mathcal{A}^T(t)]\,R_{\mathcal{G}}(t)$$

for all $t\in[\alpha_\infty,\infty)$. In particular, this means that the matrices $R_{\mathcal{G}}(t)$ and $\mathcal{A}(t)+\mathcal{A}^T(t)$ commute on $[\alpha_\infty,\infty)$, i.e., the commutator $[R_{\mathcal{G}}(t),\,\mathcal{A}(t)+\mathcal{A}^T(t)]=0$ for every $t\in[\alpha_\infty,\infty)$, showing the first part of the lemma. For the proof of the second part we note that the orthogonal projector $R_{\mathcal{G}}(t)$ solves the Riccati equation (1.2) on $[\alpha_\infty,\infty)$, by Proposition 3.9. Moreover, by using formula (4.4) and the first identity in (4.3) equation (1.2) reads on $[\alpha_\infty,\infty)$ as

$$R_{\mathcal{G}}' \overset{(1.2)}{=} AR_{\mathcal{G}} - [R_{\mathcal{G}}\,(A+A^T)\,R_{\mathcal{G}} - R_{\mathcal{G}}A^T] \overset{(4.3),(4.4)}{=} \mathcal{A}R_{\mathcal{G}} - R_{\mathcal{G}}\mathcal{A} = [\mathcal{A}, R_{\mathcal{G}}].$$

Thus, the matrix $R_{\mathcal{G}}(t)$ solves system (4.1) on $[\alpha_\infty,\infty)$. □

Remark 4.2. We remark that the formulas in (4.1) are equivalent with

$$(I-R_{\mathcal{G}})' = [\mathcal{A}(t),\, I-R_{\mathcal{G}}] = [I-R_{\mathcal{G}},\, \mathcal{A}^T(t)], \quad t\in[\alpha_\infty,\infty), \tag{4.6}$$

as one can easily check. The matrix $R_{\mathcal{G}}(t)$ satisfies on $[\alpha_\infty,\infty)$ also the relations

$$\left.\begin{aligned} R_{\mathcal{G}}'(t) = \mathcal{A}(t)\,R_{\mathcal{G}}(t) + R_{\mathcal{G}}(t)\,\mathcal{A}^T(t) - R_{\mathcal{G}}(t)\,[\mathcal{A}(t)+\mathcal{A}^T(t)]\,R_{\mathcal{G}}(t), \\ \operatorname{Im}\mathcal{B}(t) \subseteq \operatorname{Im}R_{\mathcal{G}}(t). \end{aligned}\right\} \tag{4.7}$$

Note that the first equation in (4.7) is the same as (1.2) with $\mathcal{A}(t)$ instead of $A(t)$.

Next we derive properties of the solutions of ($\mathcal{R}$), which are based on the projector $R_{\mathcal{G}}(t)$ and Lemma 4.1.

Theorem 4.3. *Let $\mathcal{G}$ be a genus of conjoined bases of* (H) *with the corresponding matrix $R_{\mathcal{G}}(t)$ in* (3.11) *and let $Q(t)$ be a solution of the Riccati equation* ($\mathcal{R}$) *on $[\alpha,\infty)\subseteq[\alpha_\infty,\infty)$. Then the matrices $R_{\mathcal{G}}(t)\,Q(t)$, $Q(t)\,R_{\mathcal{G}}(t)$, and $R_{\mathcal{G}}(t)\,Q(t)\,R_{\mathcal{G}}(t)$ also solve* ($\mathcal{R}$) *on $[\alpha,\infty)$.*

Proof. Let $R_{\mathcal{G}}(t)$ and $Q(t)$ be as in the theorem. By using ($\mathcal{R}$), the second formula in (4.1), and the identities $R_{\mathcal{G}}(t)\,\mathcal{C}(t)=\mathcal{C}(t)$ and $\mathcal{B}(t)\,R_{\mathcal{G}}(t)=\mathcal{B}(t)$ for every $t\in[\alpha,\infty)$ we obtain that

$$\begin{aligned} (R_{\mathcal{G}}\,Q)' = R_{\mathcal{G}}'\,Q + R_{\mathcal{G}}\,Q' &\overset{(4.1),(\mathcal{R})}{=} [R_{\mathcal{G}},\mathcal{A}^T]\,Q + R_{\mathcal{G}}\,(\mathcal{C} - Q\,\mathcal{A} - \mathcal{A}^T Q - Q\,\mathcal{B}\,Q) \\ &= R_{\mathcal{G}}\,\mathcal{A}^T Q - \mathcal{A}^T R_{\mathcal{G}}\,Q + \mathcal{C} - R_{\mathcal{G}}\,Q\,\mathcal{A} - R_{\mathcal{G}}\,\mathcal{A}^T Q - R_{\mathcal{G}}\,Q\,\mathcal{B}\,R_{\mathcal{G}}\,Q \\ &= -\mathcal{A}^T(R_{\mathcal{G}}\,Q) - (R_{\mathcal{G}}\,Q)\,\mathcal{A} - (R_{\mathcal{G}}\,Q)\,\mathcal{B}\,(R_{\mathcal{G}}\,Q) + \mathcal{C} \end{aligned}$$

on $[\alpha,\infty)$. Thus, the matrix $R_{\mathcal{G}}(t)\,Q(t)$ solves ($\mathcal{R}$) on $[\alpha,\infty)$. Similarly, by the first formula in (4.1) and the identities $\mathcal{C}(t)\,R_{\mathcal{G}}(t)=\mathcal{C}(t)$ and $R_{\mathcal{G}}(t)\,\mathcal{B}(t)=\mathcal{B}(t)$ for all $t\in[\alpha,\infty)$ we get

$$\begin{aligned} (QR_{\mathcal{G}})' = Q'R_{\mathcal{G}} + QR_{\mathcal{G}}' &\overset{(\mathcal{R}),(4.1)}{=} (\mathcal{C} - Q\,\mathcal{A} - \mathcal{A}^T Q - Q\,\mathcal{B}\,Q)\,R_{\mathcal{G}} + Q\,[\mathcal{A},R_{\mathcal{G}}] \\ &= \mathcal{C} - Q\,\mathcal{A}R_{\mathcal{G}} - \mathcal{A}^T QR_{\mathcal{G}} - QR_{\mathcal{G}}\,\mathcal{B}\,QR_{\mathcal{G}} + Q\,\mathcal{A}R_{\mathcal{G}} - QR_{\mathcal{G}}\mathcal{A} \\ &= -\mathcal{A}^T(QR_{\mathcal{G}}) - (QR_{\mathcal{G}})\,\mathcal{A} - (QR_{\mathcal{G}})\,\mathcal{B}\,(QR_{\mathcal{G}}) + \mathcal{C} \end{aligned}$$

on $[\alpha,\infty)$, showing that the matrix $Q(t)\,R_{\mathcal{G}}(t)$ solves ($\mathcal{R}$) on $[\alpha,\infty)$. Finally, by combining these results we get that also $R_{\mathcal{G}}(t)\,[Q(t)\,R_{\mathcal{G}}(t)] = [R_{\mathcal{G}}(t)\,Q(t)]\,R_{\mathcal{G}}(t) = R_{\mathcal{G}}(t)\,Q(t)\,R_{\mathcal{G}}(t)$ is a solution of ($\mathcal{R}$) on $[\alpha,\infty)$ and the proof is complete. ☐

Remark 4.4. The symmetry of equation ($\mathcal{R}$) implies that the matrix $Q^T(t)$ solves equation ($\mathcal{R}$) on $[\alpha,\infty)$. By applying Theorem 4.3 for $Q := Q^T$ we then obtain that also the matrices $R_{\mathcal{G}}(t)\,Q^T(t)$, $Q^T(t)\,R_{\mathcal{G}}(t)$, and $R_{\mathcal{G}}(t)\,Q^T(t)\,R_{\mathcal{G}}(t)$ are solutions of ($\mathcal{R}$) on $[\alpha,\infty)$.

Corollary 4.5. *With the assumptions and notations of Theorem 4.3, the matrix $Q(t)$ satisfies the inclusion $\operatorname{Im} Q(t) \subseteq \operatorname{Im} R_{\mathcal{G}}(t)$, resp. the inclusion $\operatorname{Im} Q^T(t) \subseteq \operatorname{Im} R_{\mathcal{G}}(t)$, for all $t \in [\alpha,\infty)$ if and only if the inclusion $\operatorname{Im} Q(t_0) \subseteq \operatorname{Im} R_{\mathcal{G}}(t_0)$, resp. the inclusion $\operatorname{Im} Q^T(t_0) \subseteq \operatorname{Im} R_{\mathcal{G}}(t_0)$, holds for some $t_0 \in [\alpha,\infty)$.*

Proof. From Theorem 4.3 and Remark 4.4 we know that the matrices $Q_*(t) := R_{\mathcal{G}}(t)\,Q(t)$ and $Q_{**}(t) := R_{\mathcal{G}}(t)\,Q^T(t)$ solve equation ($\mathcal{R}$) on $[\alpha,\infty)$. Fix $t_0 \in [\alpha,\infty)$. If $\operatorname{Im} Q(t_0) \subseteq \operatorname{Im} R_{\mathcal{G}}(t_0)$, then the matrix $Q_*(t_0) = Q(t_0)$ and by the uniqueness of solutions of ($\mathcal{R}$) we obtain the equality $Q_*(t) = Q(t)$ on $[\alpha,\infty)$. The latter identity means that the inclusion $\operatorname{Im} Q(t) \subseteq \operatorname{Im} R_{\mathcal{G}}(t)$ holds for every $t \in [\alpha,\infty)$. By using the similar arguments the relation $\operatorname{Im} Q^T(t_0) \subseteq \operatorname{Im} R_{\mathcal{G}}(t_0)$ implies that $Q_{**}(t_0) = Q(t_0)$ and consequently, we have the equality $Q_{**}(t) = Q(t)$ on $[\alpha,\infty)$. Hence, the inclusion $\operatorname{Im} Q^T(t) \subseteq \operatorname{Im} R_{\mathcal{G}}(t)$ holds for every $t \in [\alpha,\infty)$. The proof of opposite implications is trivial. ☐

For our reference we now present an auxiliary result from linear algebra about orthogonal projectors.

Lemma 4.6. *Let $Z \in \mathbb{R}^{n\times n}$ be an orthogonal projector. Then $K, L \in \mathbb{R}^{n\times n}$ satisfy*

$$\operatorname{Im} K \subseteq \operatorname{Im} Z \quad \textit{and} \quad \operatorname{Im} L \subseteq \operatorname{Ker} Z \tag{4.8}$$

if and only if $K = ZE$ and $L = (I - Z)\,E$ for some matrix $E \in \mathbb{R}^{n\times n}$. In this case the equality $\operatorname{Ker} K \cap \operatorname{Ker} L = \operatorname{Ker} E$ holds.

Proof. Let Z be as in the lemma. If the matrices K and L satisfy (4.8), then for $E := K + L$ we have that

$$ZE = ZK + ZL = K, \quad (I - Z)\,E = (I - Z)\,K + (I - Z)\,L = L.$$

The opposite implication is trivial. Finally, it is easy to see that in this case we have the equality $\operatorname{Ker} K \cap \operatorname{Ker} L = \operatorname{Ker} E$, which completes the proof. ☐

Remark 4.7. Let Z be an orthogonal projector and let K and L be matrices satisfying (4.8). The results in Lemma 4.6 then show that $\operatorname{Ker} K \cap \operatorname{Ker} L = \{0\}$ if and only if the matrix $E = K + L$ is nonsingular. In particular, in this case the inclusions in (4.8) are implemented as equalities, i.e., the identities $\operatorname{Im} K = \operatorname{Im} Z$ and $\operatorname{Im} L = \operatorname{Ker} Z$ hold. We also note that the condition $\operatorname{Ker} K \cap \operatorname{Ker} L = \{0\}$ is equivalent with $\operatorname{rank}\,(K^T,\, L^T)^T = n$.

Let $\mathcal{G}$ be a genus of conjoined bases of (H) and let $R_{\mathcal{G}}(t)$ be its representing orthogonal projector in (3.11). For a given solution $Q(t)$ of the Riccati equation ($\mathcal{R}$) on a subinterval $[\alpha,\infty) \subseteq [\alpha_\infty,\infty)$ we consider the following system of first order linear differential equations

$$\left.\begin{aligned} \Theta' &= [\mathcal{A}(t) + \mathcal{B}(t)\,Q(t)]\,\Theta, \\ \Omega' &= \mathcal{A}(t)\,\Omega + [I - R_{\mathcal{G}}(t)]\{C(t) - [A(t) + A^T(t)]\,Q(t)\}\,\Theta, \end{aligned}\right\} \tag{4.9}$$

on $[\alpha,\infty)$ together with the initial conditions

$$\Theta(\alpha)=K, \quad \Omega(\alpha)=L, \tag{4.10}$$

where the matrices $K, L\in\mathbb{R}^{n\times n}$ satisfy

$$\operatorname{Im} K\subseteq \operatorname{Im} R_{\mathcal{G}}(\alpha), \quad \operatorname{Im} L\subseteq \operatorname{Ker} R_{\mathcal{G}}(\alpha), \quad \operatorname{rank}(K^T, L^T)^T=n. \tag{4.11}$$

We will study the properties of solutions of system (4.9), which will serve for the formulation and proofs of the main results of this section. The first equation in (4.9) is motivated by the approach in [23, Chapter 2, Lemma 2.1], which is adopted here to the setting of uncontrollable systems (H).

Remark 4.8. Given a solution $Q(t)$ of ($\mathcal{R}$) on $[\alpha,\infty)$ we note that for any matrices K and L satisfying (4.11) there exist unique matrices $\Theta(t)$ and $\Omega(t)$, which solve the equations in (4.9) on $[\alpha,\infty)$ with $\Theta(\alpha)=K$ and $\Omega(\alpha)=L$. Moreover, in this case we have from Lemma 4.6 and Remark 4.7 with $Z:=R_{\mathcal{G}}(\alpha)$ that $\Theta(\alpha)=R_{\mathcal{G}}(\alpha)\,E$ and $\Omega(\alpha)=[I-R_{\mathcal{G}}(\alpha)]\,E$ for some nonsingular matrix E. These observations then imply that the initial value problem (4.9)–(4.10) with (4.11) has the solution (Θ,Ω), which is unique up to a right nonsingular multiple. More precisely, if (Θ_0,Ω_0) is another solution of (4.9)–(4.11), then there exists a constant nonsingular matrix $M\in\mathbb{R}^{n\times n}$ such that $\Theta_0(t)=\Theta(t)\,M$ and $\Omega_0(t)=\Omega(t)\,M$ for all $t\in[\alpha,\infty)$.

Proposition 4.9. *Let $\mathcal{G}$ be a genus of conjoined bases of* (H) *with the corresponding matrix $R_{\mathcal{G}}(t)$ in* (3.11)*. Moreover, let $Q(t)$ be a solution of equation* ($\mathcal{R}$) *on $[\alpha,\infty)\subseteq[\alpha_\infty,\infty)$ and let (Θ,Ω) be a solution of the associated system in* (4.9) *on $[\alpha,\infty)$. Then the matrix $\Theta(t)$ has a constant kernel on $[\alpha,\infty)$ and the matrices $V(t):=[I-R_{\mathcal{G}}(t)]\,\Theta(t)$ and $W(t):=R_{\mathcal{G}}(t)\,\Omega(t)$ solve on $[\alpha,\infty)$ the linear differential equation $Y'=\mathcal{A}(t)\,Y$. In addition, if the matrices $K:=\Theta(\alpha)$ and $L:=\Omega(\alpha)$ satisfy the conditions in* (4.11)*, then for all $t\in[\alpha,\infty)$ we have*

$$\operatorname{Im}\Theta(t)=\operatorname{Im} R_{\mathcal{G}}(t), \quad \operatorname{Im}\Omega(t)=\operatorname{Ker} R_{\mathcal{G}}(t), \quad \operatorname{rank}(\Theta^T(t),\,\Omega^T(t))^T=n. \tag{4.12}$$

Proof. Let $R_{\mathcal{G}}(t)$, $Q(t)$, $\Theta(t)$, $\Omega(t)$, K, and L be as in the proposition. By the uniqueness of solutions of the first equation in (4.9) we have that $\Theta(t)=\Phi_\alpha(t)\,\Theta(\alpha)$, where $\Phi_\alpha(t)$ is the associated fundamental matrix normalized at the point α, i.e.,

$$\Phi_\alpha'=[\mathcal{A}(t)+\mathcal{B}(t)\,Q(t)]\,\Phi_\alpha, \quad t\in[\alpha,\infty), \quad \Phi_\alpha(\alpha)=I. \tag{4.13}$$

This implies that $\operatorname{Ker}\Theta(t)=\operatorname{Ker}\Theta(\alpha)$ for every $t\in[\alpha,\infty)$, i.e., the matrix $\Theta(t)$ has constant kernel on $[\alpha,\infty)$. Next we show that the matrices $V(t)$ and $W(t)$ satisfy on $[\alpha,\infty)$ the equation $Y'=\mathcal{A}(t)\,Y$. Indeed, by using (4.1), (4.6), (4.9), and the inclusion in (4.7) we obtain on $[\alpha,\infty)$ that

$$\begin{aligned} V' &= (I-R_{\mathcal{G}})'\,\Theta+(I-R_{\mathcal{G}})\,\Theta' \overset{(4.6),(4.9)}{=} [\mathcal{A}, I-R_{\mathcal{G}}]\,\Theta+(I-R_{\mathcal{G}})\,(\mathcal{A}+\mathcal{B}\,Q)\,\Theta \\ &\overset{(4.7)}{=} \mathcal{A}\,(I-R_{\mathcal{G}})\,\Theta-(I-R_{\mathcal{G}})\,\mathcal{A}\,\Theta+(I-R_{\mathcal{G}})\,\mathcal{A}\,\Theta=\mathcal{A}\,V, \end{aligned} \tag{4.14}$$

$$\begin{aligned} W' &= R_{\mathcal{G}}'\,\Omega+R_{\mathcal{G}}\,\Omega' \overset{(4.1),(4.9)}{=} [\mathcal{A}, R_{\mathcal{G}}]\,\Omega+R_{\mathcal{G}}\{\mathcal{A}\,\Omega+(I-R_{\mathcal{G}})\,[C-(A+A^T)\,Q]\,\Theta\} \\ &= \mathcal{A}\,R_{\mathcal{G}}\,\Omega-R_{\mathcal{G}}\mathcal{A}\,\Omega+R_{\mathcal{G}}\mathcal{A}\,\Omega=\mathcal{A}\,W. \end{aligned} \tag{4.15}$$

Moreover, suppose that the matrices K and L satisfy (4.11). Then $V(\alpha)=0=W(\alpha)$, which in turn, by uniqueness of solutions of (4.14) and (4.15), implies that $V(t)=0=W(t)$ for all $t\in[\alpha,\infty)$. Therefore, we have $\operatorname{Im}\Theta(t)\subseteq\operatorname{Im} R_{\mathcal{G}}(t)$ and $\operatorname{Im}\Omega(t)\subseteq\operatorname{Ker} R_{\mathcal{G}}(t)$ on $[\alpha,\infty)$. And since the matrices $\Theta(t)$ and $R_{\mathcal{G}}(t)$ have constant ranks on $[\alpha,\infty)$ and the equality $\operatorname{rank}\Theta(\alpha)=\operatorname{rank} R_{\mathcal{G}}(\alpha)$ holds by Remark 4.8, we obtain that even $\operatorname{Im}\Theta(t) = \operatorname{Im} R_{\mathcal{G}}(t)$ for every $t \in [\alpha,\infty)$. Now we shall prove the last condition in (4.12), which is clearly equivalent with the identity $\operatorname{Ker}\Theta(t) \cap \operatorname{Ker}\Omega(t) = \{0\}$ on $[\alpha,\infty)$. Fix $\beta \in [\alpha,\infty)$ and let $v \in \operatorname{Ker}\Theta(\beta) \cap \operatorname{Ker}\Omega(\beta)$. From the fact that $\operatorname{Ker}\Theta(t)$ is constant on $[\alpha,\infty)$ it then follows that $\Theta(t)\, v = 0$ for all $t \in [\alpha,\infty)$. In particular, this means that the function $w(t) := \Omega(t)\, v$ satisfies on $[\alpha,\infty)$ the identity $w'(t) = \mathcal{A}(t)\, w(t)$, by (4.9). But $w(\beta) = 0$ and hence, by the uniqueness of solutions of the equation $y' = \mathcal{A}(t)\, y$ we get that $w(t) = 0$ for every $t \in [\alpha,\infty)$. Therefore, the vector $v \in \operatorname{Ker}\Theta(\alpha) \cap \operatorname{Ker}\Omega(\alpha) = \operatorname{Ker} K \cap \operatorname{Ker} L$, which in turn implies that $v = 0$, by (4.11). Thus, the subspace $\operatorname{Ker}\Theta(t) \cap \operatorname{Ker}\Omega(t) = \{0\}$ for all $t \in [\alpha,\infty)$. Finally, with the aid of Remark 4.7 we conclude that $\operatorname{Im}\Omega(t) = \operatorname{Ker} R_{\mathcal{G}}(t)$ on $[\alpha,\infty)$, showing the second condition in (4.12). $\square$

Remark 4.10. The results in Proposition 4.9 and Remark 2.2 imply that for any solution (Θ,Ω) of (4.9)–(4.11) the matrix $\Theta^\dagger \in \mathrm{C}^1_{\mathrm{p}}$ and satisfies the equation

$$[\Theta^\dagger(t)]' = -\Theta^\dagger(t)\, [\mathcal{A}(t) + \mathcal{B}(t)\, Q(t)\, R_{\mathcal{G}}(t)], \quad t \in [\alpha,\infty).$$

Moreover, from Remark 4.8 it follows that for a given solution $Q(t)$ of the Riccati equation ($\mathcal{R}$) on $[\alpha,\infty) \subseteq [\alpha_\infty,\infty)$ there exists the unique such a pair (Θ,Ω) satisfying $\Theta(\alpha) = R_{\mathcal{G}}(\alpha)$ and $\Omega(\alpha) = I - R_{\mathcal{G}}(\alpha)$. Obviously, in this case we have the equalities $\Theta(t) = \Phi_\alpha(t)\, R_{\mathcal{G}}(\alpha)$ and $\operatorname{Im}\Theta^\dagger(t) = \operatorname{Im}\Theta^T(t) \equiv \operatorname{Im} R_{\mathcal{G}}(\alpha)$ on $[\alpha,\infty)$, where $\Phi_\alpha(t)$ is the fundamental matrix in (4.13). In particular, the matrix $\Theta^\dagger(t)$ then satisfies for every $t \in [\alpha,\infty)$ the formula

$$\begin{aligned} \Theta^\dagger(t) &= R_{\mathcal{G}}(\alpha)\, \Theta^\dagger(t) = \Phi_\alpha^{-1}(t)\, \Phi_\alpha(t)\, R_{\mathcal{G}}(\alpha)\, \Theta^\dagger(t) \\ &= \Phi_\alpha^{-1}(t)\, \Theta(t)\, \Theta^\dagger(t) \overset{(4.12)}{=} \Phi_\alpha^{-1}(t)\, R_{\mathcal{G}}(t). \end{aligned} \tag{4.16}$$

On the other hand, by the aid of (4.12) we obtain the equality

$$\operatorname{Im}\Phi_\alpha(t)\, R_{\mathcal{G}}(\alpha) = \operatorname{Im}\Theta(t) \overset{(4.12)}{=} \operatorname{Im} R_{\mathcal{G}}(t), \quad t \in [\alpha,\infty). \tag{4.17}$$

This equality is an important property of the matrix function $\Phi_\alpha(t)$, which will be utilized in the proof of Theorem 4.16 below.

In the following remark we introduce an important matrix (called the F-matrix) in terms of a solution $\Theta(t)$ of (4.9). For an invertible $\Theta(t)$ this matrix was considered in [23, Section 2.2]. Here we allow $\Theta(t)$ to be singular.

Remark 4.11. (i) The properties of the matrix $\Theta(t)$ allow to define the function

$$F_\alpha(t) := \int_\alpha^t \Theta^\dagger(s)\, \mathcal{B}(s)\, \Theta^{\dagger T}(s)\, \mathrm{d}s, \quad t \in [\alpha,\infty), \tag{4.18}$$

which will be referred to as the *F-matrix* corresponding to the solution $Q(t)$ with respect to the genus $\mathcal{G}$. From (4.18) it immediately follows that $F_\alpha(t)$ is symmetric and the inclusion $\operatorname{Im} F_\alpha(t) \subseteq \operatorname{Im} R_{\mathcal{G}}(\alpha)$ holds for every $t \in [\alpha,\infty)$ and $F_\alpha \in \mathrm{C}^1_{\mathrm{p}}$. Moreover, under (1.1) the matrix $F_\alpha(t)$ is nonnegative definite and nondecreasing with $F_\alpha'(t) = \Theta^\dagger(t)\, \mathcal{B}(t)\, \Theta^{\dagger T}(t) \geq 0$ on $[\alpha,\infty)$. Therefore, the subspace $\operatorname{Ker} F_\alpha(t)$ is nonincreasing on $[\alpha,\infty)$, and hence eventually constant. Consequently, the properties of Moore–Penrose pseudoinverse displayed in Remark 2.2 imply that $F_\alpha^\dagger \in \mathrm{C}^1_{\mathrm{p}}$ with $(F_\alpha^\dagger)'(t) = -F_\alpha^\dagger(t)\, F_\alpha'(t)\, F_\alpha^\dagger(t) = -F_\alpha^\dagger(t)\, \Theta^\dagger(t)\, \mathcal{B}(t)\, \Theta^{\dagger T}(t)\, F_\alpha^\dagger(t) \leq 0$ for large t. Thus, the matrix $F_\alpha^\dagger(t)$ is nonincreasing for large t. And since $F_\alpha^\dagger(t)$ is nonnegative definite on $[\alpha,\infty)$, it follows that the limit of $F_\alpha^\dagger(t)$ exists as $t \to \infty$, i.e.,

$$D_\alpha := \lim_{t\to\infty} F_\alpha^\dagger(t). \tag{4.19}$$

Clearly, the matrix D_α defined in (4.19) is symmetric and nonnegative definite and the inclusion $\operatorname{Im} D_\alpha \subseteq \operatorname{Im} R_{\mathcal{G}}(\alpha)$ holds. In addition, we note that with the aid of (4.16) and the identity $R_{\mathcal{G}}(t)\, \mathcal{B}(t)\, R_{\mathcal{G}}(t) = \mathcal{B}(t)$ on $[\alpha, \infty)$ the matrix $F_\alpha(t)$ in (4.18) can be also represented as

$$\begin{aligned} F_\alpha(t) &\overset{(4.18),(4.16)}{=} \int_\alpha^t \Phi_\alpha^{-1}(s)\, R_{\mathcal{G}}(s)\, \mathcal{B}(s)\, R_{\mathcal{G}}(s)\, \Phi_\alpha^{T-1}(s)\, \mathrm{d}s \\ &= \int_\alpha^t \Phi_\alpha^{-1}(s)\, \mathcal{B}(s)\, \Phi_\alpha^{T-1}(s)\, \mathrm{d}s \end{aligned} \tag{4.20}$$

for all $t \in [\alpha, \infty)$ with $\Phi_\alpha(t)$ defined in (4.13).

(ii) There is another important property of the F-matrix introduced in (4.18). For a given genus $\mathcal{G}$ of conjoined bases of (H) with $R_{\mathcal{G}}(t)$ in (3.11) let $Q(t)$ be a solution of ($\mathcal{R}$) on $[\alpha, \infty) \subseteq [\alpha_\infty, \infty)$. By Theorem 4.3 we know that also the matrix $\tilde{Q}(t) := R_{\mathcal{G}}(t)\, Q(t)\, R_{\mathcal{G}}(t)$ solves ($\mathcal{R}$) on $[\alpha, \infty)$. Moreover, let $\Theta(t)$ and $\tilde{\Theta}(t)$ be the corresponding matrices from Remark 4.10, that is,

$$\left.\begin{aligned} \Theta' = [\mathcal{A}(t) + \mathcal{B}(t)\, Q(t)]\, \Theta, \quad \tilde{\Theta}' = [\mathcal{A}(t) + \mathcal{B}(t)\, \tilde{Q}(t)]\, \tilde{\Theta}, \quad t \in [\alpha, \infty), \\ \Theta(\alpha) = R_{\mathcal{G}}(\alpha) = \tilde{\Theta}(\alpha). \end{aligned}\right\} \tag{4.21}$$

With the aid of the identities $R_{\mathcal{G}}(t)\, \tilde{\Theta}(t) = \tilde{\Theta}(t)$ and $\mathcal{B}(t) R_{\mathcal{G}}(t) = \mathcal{B}(t)$ on $[\alpha, \infty)$ we obtain the equality

$$\begin{aligned} \tilde{\Theta}'(t) &\overset{(4.21)}{=} [\mathcal{A}(t) + \mathcal{B}(t)\, \tilde{Q}(t)]\, \tilde{\Theta}(t) = \mathcal{A}(t)\, \tilde{\Theta}(t) + \mathcal{B}(t) R_{\mathcal{G}}(t)\, Q(t)\, R_{\mathcal{G}}(t)\, \tilde{\Theta}(t) \\ &= \mathcal{A}(t)\, \tilde{\Theta}(t) + \mathcal{B}(t)\, Q(t)\, \tilde{\Theta}(t) = [\mathcal{A}(t) + \mathcal{B}(t)\, Q(t)]\, \tilde{\Theta}(t) \end{aligned}$$

for every $t \in [\alpha, \infty)$. Therefore, the matrices $\Theta(t)$ and $\tilde{\Theta}(t)$ solve the same equation on $[\alpha, \infty)$ and hence, $\Theta(t) = \tilde{\Theta}(t)$ for all $t \in [\alpha, \infty)$ by the last condition in (4.21). Consequently, the matrices $F_\alpha(t)$ and $\tilde{F}_\alpha(t)$ in (4.18) associated with the solutions $Q(t)$ and $\tilde{Q}(t)$, respectively, satisfy the equality $F_\alpha(t) = \tilde{F}_\alpha(t)$ on $[\alpha, \infty)$.

The representation of the matrix $F_\alpha(t)$ in (4.20) in terms of the fundamental matrix $\Phi_\alpha(t)$ of (4.13) allows to apply the original result in [23, Lemma 2.1, pg. 12] for symmetric solutions $Q(t)$ of ($\mathcal{R}$). This yields the following statement, which will be utilized in our further analysis.

Proposition 4.12. *Let $\mathcal{G}$ be a genus of conjoined bases of* (H) *with the matrix $R_{\mathcal{G}}(t)$ in* (3.11) *and let $Q(t)$ be a symmetric solution of the Riccati equation* ($\mathcal{R}$) *on $[\alpha, \infty) \subseteq [\alpha_\infty, \infty)$. Moreover, let $\Phi_\alpha(t)$ and $F_\alpha(t)$ be the corresponding matrices in* (4.13) *and* (4.20)*, respectively. Then an $n \times n$ matrix-valued function $\tilde{Q}(t)$ solves* ($\mathcal{R}$) *on $[\alpha, \infty)$ if and only if the constant matrix $G := \tilde{Q}(\alpha) - Q(\alpha)$ is such that the matrix $I + F_\alpha(t)\, G$ is nonsingular on $[\alpha, \infty)$ and*

$$\tilde{Q}(t) = Q(t) + \Phi_\alpha^{T-1}(t)\, G\, [I + F_\alpha(t)\, G]^{-1}\, \Phi_\alpha^{-1}(t) \quad \textit{for every } t \in [\alpha, \infty). \tag{4.22}$$

Remark 4.13. Let K be a given $n \times n$ matrix and let $\tilde{Q}(t)$ be the solution of the Riccati equation ($\mathcal{R}$) satisfying $\tilde{Q}(\alpha) = K$. From Proposition 4.12 it then follows that the matrix $\tilde{Q}(t)$ as a solution of ($\mathcal{R}$) can be extended to the whole interval $[\alpha, \infty)$ if and only if the matrix $G := K - Q(\alpha)$ is such that the matrix $I + F_\alpha(t)\, G$ is nonsingular for all $t \in [\alpha, \infty)$. In this case, the solution $\tilde{Q}(t)$ has the representation in (4.22).

Formula (4.22) allows to derive inequalities between symmetric solutions of the Riccati equation ($\mathcal{R}$). We note that the statement about the constant rank of $\tilde{Q}(t) - Q(t)$ corresponds to [23, Corollary 2, pg. 13].

Corollary 4.14. *Let $\mathcal{G}$ be a genus of conjoined bases of (H) with the corresponding matrix $R_{\mathcal{G}}(t)$ in (3.11) and let $Q(t)$ and $\tilde{Q}(t)$ be symmetric solutions of the Riccati equation ($\mathcal{R}$) on $[\alpha,\infty) \subseteq [\alpha_\infty,\infty)$. Then the quantities*

$$\operatorname{rank}[\tilde{Q}(t) - Q(t)] \text{ and } \operatorname{ind}[\tilde{Q}(t) - Q(t)] \text{ are constant on } [\alpha,\infty). \tag{4.23}$$

In particular, the inequality $\tilde{Q}(t) \geq Q(t)$ holds on $[\alpha,\infty)$ if and only if $\tilde{Q}(\alpha) \geq Q(\alpha)$, and the inequality $\tilde{Q}(t) > Q(t)$ holds on $[\alpha,\infty)$ if and only if $\tilde{Q}(\alpha) > Q(\alpha)$.

Proof. Let $Q(t)$ and $\tilde{Q}(t)$ be as in the corollary and set $G := \tilde{Q}(\alpha) - Q(\alpha)$. With the aid of formula (4.22) we then obtain that $\operatorname{rank}[\tilde{Q}(t) - Q(t)] \equiv \operatorname{rank} G = \operatorname{rank}[\tilde{Q}(\alpha) - Q(\alpha)]$ on $[\alpha,\infty)$. Moreover, the continuity of the matrices $Q(t)$ and $\tilde{Q}(t)$ implies that also the quantity $\operatorname{ind}[\tilde{Q}(t) - Q(t)]$ is constant on $[\alpha,\infty)$, completing the proof of the statements in (4.23). Finally, the assertions in the second part of the corollary follow immediately from (4.23). ☐

The next statement extends to an arbitrary genus $\mathcal{G}$ the result in [7, Corollary (iv), pp. 52–53], in which we consider one system (H).

Corollary 4.15. *Assume (1.1) and let $\mathcal{G}$ be a genus of conjoined bases of (H) with the corresponding matrix $R_{\mathcal{G}}(t)$ in (3.11). Let $Q(t)$ be a symmetric solution of the Riccati equation ($\mathcal{R}$) on $[\alpha,\infty) \subseteq [\alpha_\infty,\infty)$ and let $\tilde{Q}(t)$ be a symmetric solution of ($\mathcal{R}$) satisfying the condition $\tilde{Q}(\alpha) \geq Q(\alpha)$. Then the matrix $\tilde{Q}(t)$ solves ($\mathcal{R}$) on the whole interval $[\alpha,\infty)$ such that the inequality $\tilde{Q}(t) \geq Q(t)$ holds for all $t \in [\alpha,\infty)$.*

Proof. Let $F_\alpha(t)$ be the matrix in (4.18) associated with $Q(t)$ on $[\alpha,\infty)$ and set $G := \tilde{Q}(\alpha) - Q(\alpha)$. We will show that the matrix $I + F_\alpha(t)\, G$ is nonsingular on $[\alpha,\infty)$. Fix $t \in [\alpha,\infty)$ and let $v \in \operatorname{Ker}[I + F_\alpha(t)\, G]$, i.e., the equality $v = -F_\alpha(t)\, Gv$ holds. Since the matrix G is symmetric and satisfies $G \geq 0$ and from Remark 4.11 we know that under the Legendre condition (1.1) the matrix $F_\alpha(t)$ is nonnegative definite, we have that $0 \leq v^T Gv = -v^T G^T F_\alpha(t)\, Gv \leq 0$. Thus, $v^T Gv = 0$ and consequently, $Gv = 0$. Therefore, the vector $v = -F_\alpha(t)\, Gv = 0$ and the matrix $I + F_\alpha(t)\, G$ is nonsingular. Finally, according to Remark 4.13 and Corollary 4.14 this then means that the matrix $\tilde{Q}(t)$ solves the Riccati equation ($\mathcal{R}$) on the whole interval $[\alpha,\infty)$ and satisfies the inequality $\tilde{Q}(t) \geq Q(t)$ for every $t \in [\alpha,\infty)$. ☐

In the next result we show further properties of the solutions of the Riccati equation ($\mathcal{R}$). Namely, we characterize a certain class of the values K of the initial conditions at some point β, which guarantee that the corresponding solution $\tilde{Q}(t)$ of ($\mathcal{R}$) with $\tilde{Q}(\beta) = K$ exists on the whole interval $[\beta,\infty)$. Another interpretation of the following statement is that any symmetric solution $Q(t)$ of ($\mathcal{R}$) on $[\alpha,\infty) \subseteq [\alpha_\infty,\infty)$ satisfying inclusion (1.4) can be decomposed into the product $R_{\mathcal{G}}(t)\, \tilde{Q}(t)\, R_{\mathcal{G}}(t)$ for a suitable, in general nonsymmetric, solution $\tilde{Q}(t)$ of ($\mathcal{R}$). This result can be regarded as a partial converse to Theorem 4.3 and it will be utilized for the classification of solutions of ($\mathcal{R}$) in Remark 4.20 below.

Theorem 4.16. *Let $\mathcal{G}$ be a genus of conjoined bases of (H) and let $R_{\mathcal{G}}(t)$ be its corresponding matrix in (3.11). Moreover, let $Q(t)$ be a symmetric solution of ($\mathcal{R}$) on $[\alpha,\infty) \subseteq [\alpha_\infty,\infty)$ satisfying condition (1.4). Let $\beta \in [\alpha,\infty)$ and $K \in \mathbb{R}^{n\times n}$ be* *given and consider the solution $\tilde{Q}(t)$ of ($\mathcal{R}$) with $\tilde{Q}(\beta) = K$. Then the following statements are equivalent.*

(i) *The matrix $\tilde{Q}(t)$ solves the Riccati equation ($\mathcal{R}$) on the whole interval $[\alpha, \infty)$ such that $R_{\mathcal{G}}(t)\, \tilde{Q}(t)\, R_{\mathcal{G}}(t) = Q(t)$ holds for every $t \in [\alpha, \infty)$.*

(ii) *The matrix K satisfies the equality $R_{\mathcal{G}}(\beta)\, K R_{\mathcal{G}}(\beta) = Q(\beta)$.*

Proof. First we note that assertion (i) implies (ii) trivially. Therefore, suppose that (ii) holds, i.e., the matrix K satisfies $R_{\mathcal{G}}(\beta)\, K R_{\mathcal{G}}(\beta) = Q(\beta)$. Let $\Phi_\alpha(t)$ and $F_\alpha(t)$ be the matrices in (4.13) and (4.18) associated with $Q(t)$ and put

$$E := I - \Phi_\alpha^T(\beta)\, [K - Q(\beta)]\, \Phi_\alpha(\beta)\, F_\alpha(\beta). \tag{4.24}$$

We observe that from Remark 4.11(i), inclusion (1.4) with symmetric $Q(t)$, and (4.17) at $t = \beta$ we have

$$\left.\begin{aligned} F_\alpha(\beta) = R_{\mathcal{G}}(\alpha)\, F_\alpha(\beta), \quad R_{\mathcal{G}}(\beta)\, Q(\beta) &\overset{(1.4)}{=} Q(\beta) \overset{(1.4)}{=} Q(\beta)\, R_{\mathcal{G}}(\beta), \\ R_{\mathcal{G}}(\beta)\, \Phi_\alpha(\beta)\, R_{\mathcal{G}}(\alpha) &\overset{(4.17)}{=} \Phi_\alpha(\beta)\, R_{\mathcal{G}}(\alpha). \end{aligned}\right\} \tag{4.25}$$

Then the matrix E in (4.24) satisfies

$$\begin{aligned} R_{\mathcal{G}}(\alpha)\, E &\overset{(4.24)}{=} R_{\mathcal{G}}(\alpha) - R_{\mathcal{G}}(\alpha)\, \Phi_\alpha^T(\beta)\, [K - Q(\beta)]\, \Phi_\alpha(\beta)\, F_\alpha(\beta) \\ &\overset{(4.25)}{=} R_{\mathcal{G}}(\alpha) - R_{\mathcal{G}}(\alpha)\, \Phi_\alpha^T(\beta)\, [R_{\mathcal{G}}(\beta)\, K R_{\mathcal{G}}(\beta) - Q(\beta)]\, \Phi_\alpha(\beta)\, R_{\mathcal{G}}(\alpha)\, F_\alpha(\beta) \\ &= R_{\mathcal{G}}(\alpha). \end{aligned} \tag{4.26}$$

We will show that the matrix E is nonsingular. Let $v \in \operatorname{Ker} E$. This means according to (4.24) that

$$v = \Phi_\alpha^T(\beta)\, [K - Q(\beta)]\, \Phi_\alpha(\beta)\, F_\alpha(\beta)\, v. \tag{4.27}$$

Moreover, from (4.26) it follows that $R_{\mathcal{G}}(\alpha)\, v = R_{\mathcal{G}}(\alpha)\, E v = 0$. Combining the latter equality together with (4.27) and with the first identity in (4.25) yields

$$\begin{aligned} v &\overset{(4.27)}{=} \Phi_\alpha^T(\beta)\, [K - Q(\beta)]\, \Phi_\alpha(\beta)\, F_\alpha(\beta)\, v \\ &= \Phi_\alpha^T(\beta)\, [K - Q(\beta)]\, \Phi_\alpha(\beta)\, F_\alpha(\beta)\, R_{\mathcal{G}}(\alpha)\, v = 0, \end{aligned}$$

which proves the nonsingularity of E. In particular, formula (4.26) is then equivalent with the equality $R_{\mathcal{G}}(\alpha)\, E^{-1} = R_{\mathcal{G}}(\alpha)$. Now set

$$G := E^{-1}\, \Phi_\alpha^T(\beta)\, [K - Q(\beta)]\, \Phi_\alpha(\beta) \tag{4.28}$$

and consider the solution $Q_*(t)$ of ($\mathcal{R}$) satisfying the initial condition $Q_*(\alpha) = Q(\alpha) + G$. We shall prove that the solution $Q_*(t)$ is defined on the whole interval $[\alpha, \infty)$ such that $Q_*(\beta) = K$. First observe that with the aid of (4.24) and (4.28) we get the identity

$$\Phi_\alpha^T(\beta)\, [K - Q(\beta)]\, \Phi_\alpha(\beta) \overset{(4.28)}{=} E\, G \overset{(4.24)}{=} G - \Phi_\alpha^T(\beta)\, [K - Q(\beta)]\, \Phi_\alpha(\beta)\, F_\alpha(\beta)\, G,$$

which implies immediately the formula

$$G = \Phi_\alpha^T(\beta)\, [K - Q(\beta)]\, \Phi_\alpha(\beta)\, [I + F_\alpha(\beta)\, G]. \tag{4.29}$$

Furthermore, by using the equality $R_{\mathcal{G}}(\alpha)\, E^{-1} = R_{\mathcal{G}}(\alpha)$ and (4.25) we obtain that

$$\begin{aligned} R_{\mathcal{G}}(\alpha)\, G R_{\mathcal{G}}(\alpha) &\overset{(4.28)}{=} R_{\mathcal{G}}(\alpha)\, E^{-1}\, \Phi_\alpha^T(\beta)\, [K - Q(\beta)]\, \Phi_\alpha(\beta)\, R_{\mathcal{G}}(\alpha) \\ &= R_{\mathcal{G}}(\alpha)\, \Phi_\alpha^T(\beta)\, [K - Q(\beta)]\, \Phi_\alpha(\beta)\, R_{\mathcal{G}}(\alpha) \\ &\overset{(4.25)}{=} R_{\mathcal{G}}(\alpha)\, \Phi_\alpha^T(\beta)\, [R_{\mathcal{G}}(\beta)\, K R_{\mathcal{G}}(\beta) - Q(\beta)]\, \Phi_\alpha(\beta)\, R_{\mathcal{G}}(\alpha) \\ &= 0. \end{aligned} \tag{4.30}$$

Fix $t \in [\alpha, \infty)$ and let $v \in \operatorname{Ker}[I + F_\alpha(t)\, G]$, i.e., the equality $v = -F_\alpha(t)\, Gv$ holds. In particular, the vector $v \in \operatorname{Im} F_\alpha(t) \subseteq \operatorname{Im} R_{\mathcal{G}}(\alpha)$, by Remark 4.11(i). This means that $v = R_{\mathcal{G}}(\alpha)\, v$, which in turn together with the equality $F_\alpha(t)\, R_{\mathcal{G}}(\alpha) = F_\alpha(t)$ and equation (4.30) yields $v = -F_\alpha(t)\, Gv = -F_\alpha(t)\, R_{\mathcal{G}}(\alpha)\, G R_{\mathcal{G}}(\alpha)\, v = 0$. Thus, the matrix $I + F_\alpha(t)\, G$ is nonsingular on $[\alpha, \infty)$ and by Remark 4.13 the solution $Q_*(t)$ exists on the whole interval $[\alpha, \infty)$ such that

$$Q_*(t) = Q(t) + \Phi_\alpha^{T-1}(t)\, G\, [I + F_\alpha(t)\, G]^{-1}\, \Phi_\alpha^{-1}(t) \tag{4.31}$$

for all $t \in [\alpha, \infty)$, by (4.22). In particular, it follows for the matrix $Q_*(\beta)$ that

$$\begin{aligned} Q_*(\beta) &\overset{(4.31)}{=} Q(\beta) + \Phi_\alpha^{T-1}(\beta)\, G\, [I + F_\alpha(\beta)\, G]^{-1}\, \Phi_\alpha^{-1}(\beta) \\ &\overset{(4.29)}{=} Q(\beta) + [K - Q(\beta)] = K = \tilde{Q}(\beta). \end{aligned}$$

Therefore, by the uniqueness of solutions of ($\mathcal{R}$) the matrix $\tilde{Q}(t)$ solves ($\mathcal{R}$) on the whole interval $[\alpha, \infty)$ with $\tilde{Q}(t) = Q_*(t)$ for every $t \in [\alpha, \infty)$. In turn, the matrix $R_{\mathcal{G}}(t)\, \tilde{Q}(t)\, R_{\mathcal{G}}(t)$ is also a solution of ($\mathcal{R}$) on $[\alpha, \infty)$, by Theorem 4.3. Finally, since $R_{\mathcal{G}}(\beta)\, \tilde{Q}(\beta)\, R_{\mathcal{G}}(\beta) = R_{\mathcal{G}}(\beta)\, K R_{\mathcal{G}}(\beta) = Q(\beta)$, we conclude by using the uniqueness of solutions of ($\mathcal{R}$) once more that $R_{\mathcal{G}}(t)\, \tilde{Q}(t) R_{\mathcal{G}}(t) = Q(t)$ for all $t \in [\alpha, \infty)$. This shows (i) and the proof is complete. □

Remark 4.17. For the completeness we note that the matrix $\tilde{Q}(t)$ in Theorem 4.16 satisfies the formula

$$\tilde{Q}(t) = Q(t) + \Phi_\alpha^{T-1}(t)\, G\, [I + F_\alpha(t)\, G]^{-1}\, \Phi_\alpha^{-1}(t), \quad t \in [\alpha, \infty).$$

This follows directly from the equality $\tilde{Q}(t) = Q_*(t)$ on $[\alpha, \infty)$ and the representation of the matrix $Q_*(t)$ in (4.31).

We are now ready to formulate the main results of this section (Theorems 4.18 and 4.21), in which we connect the solutions $Q(t)$ of the Riccati equation ($\mathcal{R}$) on $[\alpha, \infty)$ with conjoined bases (X, U) with constant kernel on $[\alpha, \infty)$ from the genus $\mathcal{G}$. These results extend the well known correspondence between the symmetric solutions $Q(t)$ of (R) on $[\alpha, \infty)$ and conjoined bases (X, U) of (H) with $X(t)$ invertible on $[\alpha, \infty)$, i.e.,

$$Q(t) = U(t)\, X^{-1}(t) \quad \text{on } [\alpha, \infty) \tag{4.32}$$

to the case of possibly noninvertible $X(t)$ on $[\alpha, \infty)$.

Theorem 4.18. *Let $\mathcal{G}$ be a genus of conjoined bases of* (H) *with the orthogonal projector $R_{\mathcal{G}}(t)$ in* (3.11). *Moreover, let (X, U) be a conjoined basis of* (H) *belonging to $\mathcal{G}$ such that (X, U) has constant kernel on a subinterval $[\alpha, \infty) \subseteq [\alpha_\infty, \infty)$ and let $Q(t)$ be the corresponding Riccati quotient in* (2.7). *Then the matrix $Q(t)$ is a symmetric solution of the Riccati equation* ($\mathcal{R}$) *on $[\alpha, \infty)$ such that the condition in* (1.4) *holds and the matrices $\Theta(t)$ and $\Omega(t)$ defined by*

$$\Theta(t) := X(t), \quad \Omega(t) := U(t) - Q(t)\, X(t), \quad t \in [\alpha, \infty), \tag{4.33}$$

solve the initial value problem (4.9)–(4.10) *on* $[\alpha,\infty)$ *with* (4.11).

Proof. Let $R(t)$ be the orthogonal projector in (2.4) associated with (X,U). From Proposition 3.11 we know that $R(t)=R_{\mathcal{G}}(t)$ on $[\alpha,\infty)$. By (2.7) the matrix $Q(t)$ then satisfies the equality $Q(t)=R_{\mathcal{G}}(t)\,U(t)\,X^{\dagger}(t)$ for every $t\in[\alpha,\infty)$. Moreover, using the identities in (1.3), (3.12), and (4.2) we obtain on $[\alpha,\infty)$ that

$$\begin{aligned}
Q' &= R_{\mathcal{G}}'\,UX^{\dagger}+R_{\mathcal{G}}\,U'X^{\dagger}+R_{\mathcal{G}}\,U(X^{\dagger})'\\
&\overset{(4.2)}{=} [R_{\mathcal{G}},\mathcal{A}^T]\,UX^{\dagger}+R_{\mathcal{G}}\,(CX-A^TU)\,X^{\dagger}+R_{\mathcal{G}}\,U(X^{\dagger})'\\
&\overset{(3.12)}{=} R_{\mathcal{G}}\mathcal{A}^TUX^{\dagger}-\mathcal{A}^TR_{\mathcal{G}}\,UX^{\dagger}+R_{\mathcal{G}}\,CR_{\mathcal{G}}-R_{\mathcal{G}}\mathcal{A}^TUX^{\dagger}-R_{\mathcal{G}}\,UX^{\dagger}(\mathcal{A}+\mathcal{B}\,Q)\\
&\overset{(1.3)}{=} -Q\,\mathcal{A}-\mathcal{A}^TQ-Q\,\mathcal{B}\,Q+\mathcal{C}.
\end{aligned}$$

Thus, the matrix $Q(t)$ solves the Riccati equation ($\mathcal{R}$) on $[\alpha,\infty)$ and condition (1.4) holds. Furthermore, according to (3.12) in Proposition 3.11 the matrix $\Theta(t)=X(t)$ satisfies the first equation in (4.9) on $[\alpha,\infty)$ while applying (3.13) and ($\mathcal{R}$) yields for the matrix $\Omega(t)$ the equality

$$\begin{aligned}
\Omega'-\mathcal{A}\,\Omega &= (U-QX)'-\mathcal{A}\,(U-QX)\\
&\overset{(3.13)}{=} [C-(A+A^T)\,Q]\,X-Q'X-QX'+\mathcal{A}\,QX\\
&\overset{(\mathcal{R}),\,(3.12)}{=} [C-(A+A^T)\,Q]\,X+(\mathcal{A}+\mathcal{A}^T)\,QX-\mathcal{C}X
\end{aligned}\tag{4.34}$$

on $[\alpha,\infty)$. Moreover, by using (4.5), (1.3), and the equalities $R_{\mathcal{G}}(t)\,Q(t)=Q(t)$ and $R_{\mathcal{G}}(t)\,X(t)=R_{\mathcal{G}}(t)$ for every $t\in[\alpha,\infty)$ the last two terms in (4.34) become

$$\begin{aligned}
(\mathcal{A}+\mathcal{A}^T)\,QX-\mathcal{C}X &\overset{(1.3)}{=} (\mathcal{A}+\mathcal{A}^T)\,R_{\mathcal{G}}\,QX-R_{\mathcal{G}}\,CR_{\mathcal{G}}X\\
&\overset{(4.5)}{=} R_{\mathcal{G}}\,(A+A^T)\,R_{\mathcal{G}}\,QX-R_{\mathcal{G}}\,CX\\
&= R_{\mathcal{G}}\,(A+A^T)\,QX-R_{\mathcal{G}}\,CX\\
&= -R_{\mathcal{G}}\,[C-(A+A^T)\,Q]\,X
\end{aligned}\tag{4.35}$$

on $[\alpha,\infty)$. By combining formulas (4.34) and (4.35) we obtain that

$$\Omega'-\mathcal{A}\,\Omega=(I-R_{\mathcal{G}})\,[C-(A+A^T)\,Q]\,X=(I-R_{\mathcal{G}})\,[C-(A+A^T)\,Q]\,\Theta$$

on $[\alpha,\infty)$, showing the second equation in (4.9). Finally, from the first identity in (4.33) we have that $\operatorname{Im}\Theta(\alpha)=\operatorname{Im}X(\alpha)=\operatorname{Im}R_{\mathcal{G}}(\alpha)$, while the second one together with the last formula in (2.8) give

$$R_{\mathcal{G}}(\alpha)\,\Omega(\alpha)=R_{\mathcal{G}}(\alpha)\,U(\alpha)-R_{\mathcal{G}}(\alpha)\,Q(\alpha)\,X(\alpha)=R_{\mathcal{G}}(\alpha)\,U(\alpha)-Q(\alpha)\,X(\alpha)=0.$$

Hence, the inclusion $\operatorname{Im}\Omega(\alpha)\subseteq\operatorname{Ker}R_{\mathcal{G}}(\alpha)$ holds. On the other hand, with the aid of (4.33) and the fact that (X,U) is a conjoined basis one can easily check that $\operatorname{Ker}\Theta(\alpha)\cap\operatorname{Ker}\Omega(\alpha)=\operatorname{Ker}X(\alpha)\cap\operatorname{Ker}U(\alpha)=\{0\}$, which is equivalent with the equality $\operatorname{rank}(\Theta^T(\alpha),\,\Omega^T(\alpha))^T=n$. Therefore, the matrices $\Theta(\alpha)$ and $\Omega(\alpha)$ satisfy the conditions in (4.11) and the proof is complete. $\square$

Remark 4.19. Let $S_\alpha(t)$ be the matrix in (2.11) associated with the conjoined basis (X,U) on $[\alpha,\infty)$ and let $F_\alpha(t)$ be the matrix in (4.18), which corresponds to the Riccati quotient $Q(t)$ on $[\alpha,\infty)$. The identities in (3.14) and (4.20) then give

$$\begin{aligned} S_\alpha(t) &= \int_\alpha^t X^\dagger(s)\,B(s)\,X^{\dagger T}(s)\,\mathrm{d}s \overset{(3.14)}{=} \int_\alpha^t X^\dagger(\alpha)\,\Phi_\alpha^{-1}(s)\,B(s)\,\Phi_\alpha^{T-1}(s)\,X^{\dagger T}(\alpha)\,\mathrm{d}s \\ &\overset{(1.3)}{=} X^\dagger(\alpha)\int_\alpha^t \Phi_\alpha^{-1}(s)\,\mathcal{B}(s)\,\Phi_\alpha^{T-1}(s)\,\mathrm{d}s\,X^{\dagger T}(\alpha) \\ &\overset{(4.20)}{=} X^\dagger(\alpha)\,F_\alpha(t)\,X^{\dagger T}(\alpha) \end{aligned} \tag{4.36}$$

for all $t\in[\alpha,\infty)$, where $\Phi_\alpha(t)$ is the fundamental matrix in (4.13). On the other hand, with the aid of the equalities $X(\alpha)\,X^\dagger(\alpha)=R_{\mathcal{G}}(\alpha)=X^{\dagger T}(\alpha)\,X^T(\alpha)$ and $F_\alpha(t)=R_{\mathcal{G}}(\alpha)\,F_\alpha(t)\,R_{\mathcal{G}}(\alpha)$ on $[\alpha,\infty)$ expression (4.36) yields the formula

$$\begin{aligned} F_\alpha(t) &= R_{\mathcal{G}}(\alpha)\,F_\alpha(t)\,R_{\mathcal{G}}(\alpha) = X(\alpha)\,X^\dagger(\alpha)\,F_\alpha(t)\,X^{\dagger T}(\alpha)\,X^T(\alpha) \\ &\overset{(4.36)}{=} X(\alpha)\,S_\alpha(t)\,X^T(\alpha). \end{aligned} \tag{4.37}$$

for every on $t\in[\alpha,\infty)$. Furthermore, let P, $P_{\mathcal{S}_\alpha}(t)$, and $P_{\mathcal{S}_\alpha\infty}$ be the matrices in (2.4) and (2.12) associated with (X,U). By combining (4.37) with the identities $X^T(\alpha)\,X^{\dagger T}(\alpha)=P$ and $S_\alpha(t)\,P=S_\alpha(t)$ we get $F_\alpha(t)\,X^{\dagger T}(\alpha)=X(\alpha)\,S_\alpha(t)\,P=X(\alpha)\,S_\alpha(t)$, which in turn through (2.24) implies that

$$\operatorname{Im} F_\alpha(t)=\operatorname{Im} X(\alpha)\,S_\alpha(t) \overset{(2.12)}{=} \operatorname{Im} X(\alpha)\,P_{\mathcal{S}_\alpha}(t) \overset{(2.24)}{=} \big(\Lambda_\alpha[\alpha,t]\big)^\perp, \tag{4.38}$$

where the subspace $\Lambda_\alpha[\alpha,t]$ is defined in Section 2. Note that equality (4.38) is in a full agreement with the monotonicity of the subspace $\operatorname{Ker} F_\alpha(t)$ in Remark 4.11(i). Moreover, by using the relation in (2.22) we obtain

$$\operatorname{Im} F_\alpha(t) \overset{(4.38)}{=} \big(\Lambda_\alpha[\alpha,t]\big)^\perp \overset{(2.22)}{\equiv} \big(\Lambda_\alpha[\alpha,\infty)\big)^\perp \quad \text{on } (\tau_{\alpha,\infty},\infty), \tag{4.39}$$

where the point $\tau_{\alpha,\infty}$ is defined in (2.23).

Remark 4.20. Based on Theorem 4.18, the result in Theorem 4.16 enables to determine all the solutions $Q(t)$ of the Riccati equation $(\mathcal{R})$ on $[\alpha,\infty)$, for which the matrix $R_{\mathcal{G}}(t)\,Q(t)\,R_{\mathcal{G}}(t)$ is the (symmetric) Riccati quotient associated with the conjoined basis (X,U) on $[\alpha,\infty)$. More precisely, if $\beta\in[\alpha,\infty)$ is a given point and $Q(t)$ is a solution of $(\mathcal{R})$ defined on a neighborhood of β, then the matrix $Q(t)$ solves $(\mathcal{R})$ on the whole interval $[\alpha,\infty)$ and the matrix $R_{\mathcal{G}}(t)\,Q(t)\,R_{\mathcal{G}}(t)$ is the Riccati quotient associated with (X,U) for every $t\in[\alpha,\infty)$ if and only if the matrix $R_{\mathcal{G}}(\beta)\,Q(\beta)\,R_{\mathcal{G}}(\beta)$ is the Riccati quotient for (X,U) at the point β. In addition, from Remarks 4.11(ii) and 4.19 it follows that the matrix $F_\alpha(t)$ in (4.18), which corresponds to every such a solution $Q(t)$, satisfies formulas (4.36)–(4.39).

Theorem 4.21. *Let $\mathcal{G}$ be a genus of conjoined bases of* (H) *with the orthogonal projector $R_{\mathcal{G}}(t)$ in* (3.11) *and let $Q(t)$ be a solution of the Riccati equation $(\mathcal{R})$ on $[\alpha,\infty)\subseteq[\alpha_\infty,\infty)$ such that the matrix $R_{\mathcal{G}}(t)\,Q(t)\,R_{\mathcal{G}}(t)$ is symmetric on $[\alpha,\infty)$. Moreover, let (Θ,Ω) be a solution of* (4.9)–(4.11) *on $[\alpha,\infty)$ and define the matrices*

$$X(t):=\Theta(t),\quad U(t):=Q(t)\,\Theta(t)+\Omega(t),\quad t\in[\alpha,\infty). \tag{4.40}$$

Then the following statements hold.

(i) *The pair (X,U) is a conjoined basis of* (H) *such that (X,U) has a constant kernel on $[\alpha,\infty)$ and belongs to the genus $\mathcal{G}$.*

(ii) *The matrix $R_{\mathcal{G}}(t)\,Q(t)\,R_{\mathcal{G}}(t)$ is the Riccati quotient in* (2.7) *associated with (X,U) on $[\alpha,\infty)$, i.e., the equality $R_{\mathcal{G}}(t)\,Q(t)\,R_{\mathcal{G}}(t)=R(t)\,U(t)\,X^\dagger(t)$ holds for all $t\in[\alpha,\infty)$, where $R(t)$ is the corresponding projector in* (2.4).

Proof. (i) First we show that the pair (X,U) is a solution of (H) on $[\alpha,\infty)$. From Proposition 4.9 we know that the matrix $\Theta(t)$ has constant kernel on $[\alpha,\infty)$ and $\operatorname{Im}\Theta(t)=\operatorname{Im}R_{\mathcal{G}}(t)$ for every $t\in[\alpha,\infty)$. Thus, the matrix $X(t)$ has constant kernel on $[\alpha,\infty)$ and $\operatorname{Im}X(t)=R_{\mathcal{G}}(t)$ for all $t\in[\alpha,\infty)$. And since $\Theta(t)$ solves the first equation in (4.9) on $[\alpha,\infty)$, we have that

$$X'(t)=[\mathcal{A}(t)+\mathcal{B}(t)\,Q(t)]\,X(t),\quad t\in[\alpha,\infty). \tag{4.41}$$

Moreover, by using (1.3), (4.40), and the inclusion in (4.7) we obtain the formula

$$B(t)\,U(t)\overset{(1.3),\,(4.40),\,(4.7)}{=}\mathcal{B}(t)\,Q(t)\,X(t)+\mathcal{B}(t)\,R_{\mathcal{G}}(t)\,\Omega(t)=\mathcal{B}(t)\,Q(t)\,X(t) \tag{4.42}$$

for every $t\in[\alpha,\infty)$. Combining identities (4.41) and (4.42) with (4.3) and the equality $R_{\mathcal{G}}(t)\,X(t)=X(t)$ for every $t\in[\alpha,\infty)$ then yields on $[\alpha,\infty)$ that

$$X'\overset{(4.41)}{=}\mathcal{A}\,X+\mathcal{B}\,Q X\overset{(4.42)}{=}\mathcal{A}\,R_{\mathcal{G}}X+BU\overset{(4.3)}{=}AR_{\mathcal{G}}X+BU=AX+BU. \tag{4.43}$$

Next we derive some additional properties of the matrices $\Theta(t)$ and $\Omega(t)$, which will simplify our calculations. In particular, by ($\mathcal{R}$) and the first equation in (4.9) we have on $[\alpha,\infty)$ that

$$(Q\,\Theta)'\overset{(\mathcal{R}),\,(4.9)}{=}(\mathcal{C}-Q\,\mathcal{A}-\mathcal{A}^T Q-Q\,\mathcal{B}\,Q)\,\Theta+Q\,(\mathcal{A}+\mathcal{B}\,Q)\,\Theta=(\mathcal{C}-\mathcal{A}^T Q)\,\Theta. \tag{4.44}$$

On the other hand, by (1.3) and the identities $R_{\mathcal{G}}(t)\,\Omega(t)=0$ and $R_{\mathcal{G}}(t)\,\Theta(t)=\Theta(t)$ for all $t\in[\alpha,\infty)$, the second equation in (4.9) reads on $[\alpha,\infty)$ as

$$\begin{aligned}\Omega'&\overset{(4.9),\,(1.3)}{=}[AR_{\mathcal{G}}-A^T(I-R_{\mathcal{G}})]\,\Omega+[C-(A+A^T)\,Q]\,\Theta-R_{\mathcal{G}}\,[C-(A+A^T)\,Q]\,\Theta\\ &=-A^T\Omega+[C-A^T Q]\,\Theta-A\,Q\,\Theta-R_{\mathcal{G}}\,C\,R_{\mathcal{G}}\Theta+R_{\mathcal{G}}\,(A+A^T)\,Q\,\Theta\\ &\overset{(1.3)}{=}-A^T\Omega+[C-A^T Q]\,\Theta-\mathcal{C}\,\Theta+[R_{\mathcal{G}}\,(A+A^T)-A]Q\,\Theta\\ &\overset{(1.3)}{=}-A^T\Omega+[C-A^T Q]\,\Theta-[\mathcal{C}-\mathcal{A}^T Q]\,\Theta.\end{aligned} \tag{4.45}$$

Now by using (4.44) and (4.45) we obtain that the matrix $U(t)$ in (4.40) satisfies

$$\begin{aligned}U'&=(Q\,\Theta)'+\Omega'=(\mathcal{C}-\mathcal{A}^T Q)\,\Theta-\mathcal{A}^T\Omega+[C-A^T Q]\,\Theta-[\mathcal{C}-\mathcal{A}^T Q]\,\Theta\\ &=C\Theta-A^T(Q\,\Theta+\Omega)\overset{(4.40)}{=}CX-A^T U\quad\text{on }[\alpha,\infty).\end{aligned} \tag{4.46}$$

Hence, equalities (4.43) and (4.46) show that the pair (X,U) solves system (H) on $[\alpha,\infty)$. Moreover, the matrix

$$X^T(t)\,U(t)=\Theta^T(t)\,Q(t)\,\Theta(t)+\Theta^T(t)\,\Omega(t)=\Theta^T(t)\,R_{\mathcal{G}}(t)\,Q(t)\,R_{\mathcal{G}}(t)\,\Theta(t)$$

is symmetric and the subspace $\operatorname{Ker}X(t)\cap\operatorname{Ker}U(t)=\operatorname{Ker}\Theta(t)\cap\operatorname{Ker}\Omega(t)=\{0\}$ for every $t\in[\alpha,\infty)$, both by Proposition 4.9. Therefore, the solution (X,U) is a conjoined basis with constant kernel on $[\alpha,\infty)$. And since the equality $\operatorname{Im}X(t)=\operatorname{Im}R_{\mathcal{G}}(t)$ holds for every $t\in[\alpha,\infty)$, we conclude that (X,U) belongs to the genus $\mathcal{G}$. For the proof of part (ii) we note that the orthogonal projector $R(t)$ in (2.4) associated with (X,U) satisfies $R(t)=R_{\mathcal{G}}(t)$ for all $t\in[\alpha,\infty)$. In particular, by using the identities $\Theta(t)\,\Theta^\dagger(t)=R_{\mathcal{G}}(t)$ and $R_{\mathcal{G}}(t)\,\Omega(t)=0$ on $[\alpha,\infty)$ we obtain that

$$R(t)\,U(t)\,X^\dagger(t)\overset{(4.40)}{=}R_{\mathcal{G}}(t)\,Q(t)\,\Theta(t)\,\Theta^\dagger(t)+R_{\mathcal{G}}(t)\,\Omega(t)\,\Theta^\dagger(t)=R_{\mathcal{G}}(t)\,Q(t)\,R_{\mathcal{G}}(t)$$

for every $t\in[\alpha,\infty)$. Thus, the matrix $R_{\mathcal{G}}(t)\,Q(t)\,R_{\mathcal{G}}(t)$ is the Riccati quotient in (2.7) associated with (X,U) on $[\alpha,\infty)$ and the proof is complete. ∎

Remark 4.22. We note that the matrices $F_\alpha(t)$ and $S_\alpha(t)$ in (4.18) and (2.11) associated with the matrix $Q(t)$ and the conjoined basis (X, U) in Theorem 4.21, respectively, satisfy the identities in (4.36) and (4.37). This follows directly from Theorem 4.18 and Remark 4.11(ii).

Remark 4.23. Let $\mathcal{G}$ be a genus of conjoined basis of (H) with the associated matrix $R_{\mathcal{G}}(t)$ in (3.11) and let $[\alpha, \infty) \subseteq [\alpha_\infty, \infty)$ be a given interval. The results in Theorems 4.18 and 4.21 provide a correspondence between the set of all conjoined basis (X, U) of (H) with constant kernel on $[\alpha, \infty)$, which belong to the genus $\mathcal{G}$, and the set of all symmetric solutions $Q(t)$ of the Riccati equation $(\mathcal{R})$ on $[\alpha, \infty)$ satisfying condition (1.4). More precisely, for every such a conjoined basis (X, U) its Riccati quotient $Q(t)$ in (2.7) is a symmetric solution of $(\mathcal{R})$ on $[\alpha, \infty)$ with $\operatorname{Im} Q(t) \subseteq \operatorname{Im} R_{\mathcal{G}}(t)$ for all $t \in [\alpha, \infty)$, as we claim in Theorem 4.18. Conversely, if $Q(t)$ is a symmetric solution of $(\mathcal{R})$ on $[\alpha, \infty)$ such that $\operatorname{Im} Q(t) \subseteq \operatorname{Im} R_{\mathcal{G}}(t)$ for every $t \in [\alpha, \infty)$, then there exists a conjoined basis (X, U) of (H) from the genus $\mathcal{G}$ with constant kernel on $[\alpha, \infty)$ such that the matrix $Q(t)$ is its corresponding Riccati quotient in (2.7), by Theorem 4.21, and the equality $R_{\mathcal{G}}(t)\, Q(t)\, R_{\mathcal{G}}(t) = Q(t)$ on $[\alpha, \infty)$. In addition, every such a conjoined basis (X, U) has the form of (4.40) for some solution (Θ, Ω) of (4.9)–(4.11) on $[\alpha, \infty)$. Finally, the observations in Remark 4.8 then imply that the conjoined basis (X, U) is uniquely determined up to a right nonsingular multiple by the genus $\mathcal{G}$ and the matrix $Q(t)$.

Remark 4.24. The representation of the solution $Q(t)$ of (R) in (4.32) corresponds to the results in Theorems 4.18 and 4.21 with the maximal genus $\mathcal{G} = \mathcal{G}_{\max}$. In this case $R_{\mathcal{G}}(t) \equiv I$, and (1.3) yields that the Riccati equations (R) and $(\mathcal{R})$ coincide.

5. **Inequalities for Riccati quotients in given genus.** In this section we derive a mutual representation of the Riccati quotients corresponding to conjoined bases of (H) from a given genus $\mathcal{G}$ (Theorem 5.3). This representation is then utilized for obtaining inequalities between two Riccati quotients (Corollary 5.5). The results presented in this section essentially generalize the discussion in [7, pg. 54] to possibly uncontrollable systems (H).

First we prove an auxiliary property of the image of the matrix $F_\alpha(t)$ in (4.18).

Lemma 5.1. *Assume* (1.1). *Let $\mathcal{G}$ be a genus of conjoined bases of* (H) *with the matrix $R_{\mathcal{G}}(t)$ in* (3.11) *and let $Q(t)$ be a solution of the Riccati equation* $(\mathcal{R})$ *on $[\alpha, \infty) \subseteq [\alpha_\infty, \infty)$ such that the matrix $R_{\mathcal{G}}(t)\, Q(t)\, R_{\mathcal{G}}(t)$ is symmetric on $[\alpha, \infty)$. Moreover, let $F_\alpha(t)$ be the matrix in* (4.18)*, which corresponds to $Q(t)$, and $R_{\Lambda\infty}(t)$ be the orthogonal projector defined in* (3.7). *Then*

$$\operatorname{Im} F_\alpha(t) \subseteq \operatorname{Im} R_{\Lambda\infty}(\alpha) \quad \textit{for every } t \in [\alpha, \infty). \tag{5.1}$$

Proof. Let (X, U) be a conjoined basis of (H) from the genus $\mathcal{G}$, which corresponds to the matrix $Q(t)$ through Theorem 4.21. It follows that (X, U) has constant kernel on $[\alpha, \infty)$ and the matrix $X(t)$ satisfies the equality $\operatorname{Im} X(t) = \operatorname{Im} R_{\mathcal{G}}(t)$ for every $t \in [\alpha, \infty)$. Moreover, the matrix $R_{\mathcal{G}}(t)\, Q(t)\, R_{\mathcal{G}}(t)$ is the Riccati quotient in (2.7) associated with (X, U) on $[\alpha, \infty)$, by Theorem 4.21(ii). Let $S_\alpha(t)$ be the S-matrix in (2.11) corresponding to the conjoined basis (X, U) on $[\alpha, \infty)$. From Remark 4.11(ii) we know that $F_\alpha(t)$ is the F-matrix in (4.18) associated with $R_{\mathcal{G}}(t)\, Q(t)\, R_{\mathcal{G}}(t)$. Thus, by combining (3.8) and (4.39) we obtain the identity

$$\operatorname{Im} F_\alpha(t) \overset{(4.39)}{\equiv} \big(\Lambda_\alpha[\alpha, \infty]\big)^\perp \overset{(3.8)}{=} \operatorname{Im} R_{\Lambda\infty}(\alpha) \quad \text{on } (\tau_{\alpha,\infty}, \infty), \tag{5.2}$$

where the point $\tau_{\alpha,\infty}$ is defined in (2.23). And since by Remark 4.11(i) the subspace $\operatorname{Im} F_\alpha(t)$ is nondecreasing on $[\alpha,\infty)$, the inclusion in (5.1) now immediately follows from (5.2). The proof is complete. $\square$

Remark 5.2. (i) Let $S_\alpha(t)$ and $P_{\mathcal{S}_\alpha\infty}$ be the matrices in (2.11) and (2.12) which correspond to the conjoined basis (X,U) on $[\alpha,\infty)\subseteq[\alpha_\infty,\infty)$. From (2.25) and (5.2) it follows that

$$\operatorname{Im} S_\alpha(t) = \operatorname{Im} P_{\mathcal{S}_\alpha\infty}, \quad \operatorname{Im} F_\alpha(t) = \operatorname{Im} R_{\Lambda\infty}(\alpha), \tag{5.3}$$

$$\operatorname{rank} S_\alpha(t) = \operatorname{rank} P_{\mathcal{S}_\alpha\infty} = n - d[\alpha,\infty) = \operatorname{rank} R_{\Lambda\infty}(\alpha) = \operatorname{rank} F_\alpha(t) \tag{5.4}$$

on $(\tau_{\alpha,\infty},\infty)$. Moreover, the matrices $S_\alpha(t)$ and $F_\alpha(t)$ satisfies the identities

$$\left.\begin{aligned} S_\alpha^\dagger(t) &= P_{\mathcal{S}_\alpha\infty}\, X^T(\alpha)\, F_\alpha^\dagger(t)\, X(\alpha)\, P_{\mathcal{S}_\alpha\infty}, \\ F_\alpha^\dagger(t) &= R_{\Lambda\infty}(\alpha)\, X^{\dagger T}(\alpha)\, S_\alpha^\dagger(t)\, X^\dagger(\alpha)\, R_{\Lambda\infty}(\alpha) \end{aligned}\right\} \tag{5.5}$$

for every $t\in(\tau_{\alpha,\infty},\infty)$, which can verify by direct computation with the aid of (3.9)–(3.10), (4.36), (4.37), and (5.3).

(ii) Furthermore, upon taking $t\to\infty$ in (5.5) we obtain the formulas

$$\left.\begin{aligned} T_\alpha &= P_{\mathcal{S}_\alpha\infty}\, X^T(\alpha)\, D_\alpha\, X(\alpha)\, P_{\mathcal{S}_\alpha\infty}, \\ D_\alpha &= R_{\Lambda\infty}(\alpha)\, X^{\dagger T}(\alpha)\, T_\alpha\, X^\dagger(\alpha)\, R_{\Lambda\infty}(\alpha), \end{aligned}\right\} \tag{5.6}$$

where the matrices T_α and D_α are defined in (2.13) and (4.19), respectively. The equalities in (5.6) then yield that $\operatorname{Im} D_\alpha\subseteq \operatorname{Im} R_{\Lambda\infty}(\alpha)$ and $\operatorname{rank} T_\alpha = \operatorname{rank} D_\alpha$. In particular, combining the last formula with Remark 2.8(ii) and the fact that $d[\alpha,\infty)=d_\infty$ implies that $\operatorname{rank} D_\beta$ is constant with respect to $\beta\in[\alpha,\infty)$.

In the following main result of this section we present a representation of two Riccati quotients corresponding to two conjoined bases of (H) from a genus $\mathcal{G}$. This result will be utilized in the classification of all distinguished solutions of ($\mathcal{R}$) at infinity in Section 7. When $\mathcal{G}=\mathcal{G}_{\max}$ is the maximal genus (in particular, when system (H) is controllable), this representation coincides with the statement in Proposition 4.12. We note that for a given genus $\mathcal{G}$ we now compare those solutions $Q(t)$ and $\tilde Q(t)$ of ($\mathcal{R}$), which are Riccati quotients according to their definition in (2.7). However, the Riccati equation ($\mathcal{R}$) may also have other solutions, which are not of this particular form.

Theorem 5.3. *Let $\mathcal{G}$ be a genus of conjoined bases of* (H) *and let (X,U) and $(\tilde X,\tilde U)$ be two conjoined bases of* (H) *with constant kernel on $[\alpha,\infty)\subseteq[\alpha_\infty,\infty)$ belonging to $\mathcal{G}$. Moreover, let P and $S_\alpha(t)$ be the matrices in* (2.5) *and* (2.11) *associated with (X,U). Suppose that $(\tilde X,\tilde U)$ is expressed in terms of (X,U) via matrices M and N as in Proposition 2.10. Then the Riccati quotients $Q(t)$ and $\tilde Q(t)$ in* (2.7) *corresponding to (X,U) and $(\tilde X,\tilde U)$, respectively, satisfy*

$$\tilde Q(t) = Q(t) + X^{\dagger T}(t)\, N[PM + S_\alpha(t)\, N]^\dagger\, X^\dagger(t), \quad t\in[\alpha,\infty). \tag{5.7}$$

Proof. Let $R_{\mathcal{G}}(t)$ be the orthogonal projector in (3.11) and let $R(t)$ and $\tilde R(t)$ be the matrices in (2.4), which correspond to the conjoined bases (X,U) and $(\tilde X,\tilde U)$, respectively. According to Proposition 3.11 and the second identity in (2.4) we then have the equalities

$$X(t)\, X^\dagger(t) \overset{(2.4)}{=} R(t) = R_{\mathcal{G}}(t) = \tilde R(t) \overset{(2.4)}{=} \tilde X(t)\, \tilde X^\dagger(t) \tag{5.8}$$

for all $t \in [\alpha, \infty)$. Moreover, the symmetry of the matrices $Q(t)$, $\tilde{Q}(t)$, $R(t)$, and $\tilde{R}(t)$ on $[\alpha, \infty)$, the fact that the matrix N is the Wronskian of (X, U) and $(\tilde{X}, \tilde{U})$ by Remark 2.11(i), and the equations in (5.8) imply that (we omit the argument t)

$$\begin{aligned} \tilde{Q} - Q &\overset{(2.7)}{=} \tilde{R}\tilde{U}\tilde{X}^{\dagger} - RUX^{\dagger} \overset{(5.8)}{=} R\tilde{U}\tilde{X}^{\dagger} - X^{\dagger T}U^{T}\tilde{R} \\ &\overset{(2.4)}{=} X^{\dagger T}X^{T}\tilde{U}\tilde{X}^{\dagger} - X^{\dagger T}\, U^{T}\tilde{X}\tilde{X}^{\dagger} \\ &= X^{\dagger T}(X^{T}\tilde{U} - U^{T}\tilde{X})\, \tilde{X}^{\dagger} = X^{\dagger T}N\tilde{X}^{\dagger} \end{aligned} \tag{5.9}$$

on $[\alpha, \infty)$. Finally, inserting the expression for the matrix $\tilde{X}^{\dagger}(t)$ in (2.37) into the equality in (5.9) yields formula (5.7) on $[\alpha, \infty)$ and the proof is complete. □

Remark 5.4. By substituting the matrix $X^{\dagger}(t)$ instead of $\tilde{X}^{\dagger}(t)$ in (5.9) we get another formula for the difference $\tilde{Q}(t) - Q(t)$. Namely, inserting the second identity in (2.37) into (5.9) and using the equality $PN = N$ and the symmetry of $S_{\alpha}(t)$ on $[\alpha, \infty)$ yields the formula

$$\begin{aligned} \tilde{Q}(t) - Q(t) &= \tilde{X}^{\dagger T}(t)\, [PM + S_{\alpha}(t)\, N]^{T} N\tilde{X}^{\dagger}(t) \\ &= \tilde{X}^{\dagger T}(t)\, [M^{T}N + N^{T}S_{\alpha}(t)\, N]\, \tilde{X}^{\dagger}(t) \end{aligned} \tag{5.10}$$

for every $t \in [\alpha, \infty)$. In addition, if $\tilde{P}$ is the projector in (2.5) associated with the conjoined basis $(\tilde{X}, \tilde{U})$, then by the identities $X^{T}\tilde{X}^{\dagger T} = \tilde{P} = \tilde{X}^{\dagger}\tilde{X}$ on $[\alpha, \infty)$, $N\tilde{P} = N$, and $M^{T}N = N^{T}M$ formula (5.10) implies (suppressing the argument t)

$$\begin{aligned} \tilde{X}^{T}[\tilde{Q} - Q]\, \tilde{X} &\overset{(5.10)}{=} \tilde{X}^{T}\tilde{X}^{\dagger T}[M^{T}N + N^{T}S_{\alpha}\, N]\, \tilde{X}^{\dagger}\tilde{X} \\ &= \tilde{P}\, [M^{T}N + N^{T}S_{\alpha}\, N]\, \tilde{P} = M^{T}N + N^{T}S_{\alpha}\, N \end{aligned} \tag{5.11}$$

on $[\alpha, \infty)$. Moreover, from (5.10) and (5.11) it immediately follows that

$$\left. \begin{aligned} \operatorname{rank}\, [\tilde{Q}(t) - Q(t)] &= \operatorname{rank}\, [M^{T}N + N^{T}S_{\alpha}(t)\, N], \\ \operatorname{ind}\, [\tilde{Q}(t) - Q(t)] &= \operatorname{ind}\, [M^{T}N + N^{T}S_{\alpha}(t)\, N] \end{aligned} \right\} \tag{5.12}$$

for every $t \in [\alpha, \infty)$. In particular, since $S_{\alpha}(\alpha) = 0$, by evaluating (5.12) at $t = \alpha$ we obtain the equalities

$$\operatorname{rank}\, [\, \tilde{Q}(\alpha) - Q(\alpha)] = \operatorname{rank} M^{T}N, \quad \operatorname{ind}\, [\, \tilde{Q}(\alpha) - Q(\alpha)] = \operatorname{ind} M^{T}N. \tag{5.13}$$

Formula (5.7) in Theorem 5.3 yields the following inequalities between two Riccati quotients associated with two conjoined bases from the genus $\mathcal{G}$.

Corollary 5.5. *With the assumptions and notation of Theorem 5.3, the Riccati quotients $Q(t)$ and $\tilde{Q}(t)$ satisfy the formulas*

$$\operatorname{rank}\, [\, \tilde{Q}(t) - Q(t)] \equiv \operatorname{rank} N, \quad \operatorname{ind}\, [\, \tilde{Q}(t) - Q(t)] \equiv \operatorname{ind} NM^{-1} \tag{5.14}$$

on $[\alpha, \infty)$. Moreover, the following statements hold.

(i) *The inequality $\tilde{Q}(t) \geq Q(t)$ holds for all $t \in [\alpha, \infty)$ if and only if $NM^{-1} \geq 0$.*
(ii) *The inequality $\tilde{Q}(t) \leq Q(t)$ holds for all $t \in [\alpha, \infty)$ if and only if $NM^{-1} \leq 0$.*
(iii) *The inequality $\tilde{Q}(t) > Q(t)$, resp. $\tilde{Q}(t) < Q(t)$, holds on the subspace $\operatorname{Im} R_{\mathcal{G}}(t)$ for all $t \in [\alpha, \infty)$ if and only if the inequality $NM^{-1} > 0$, resp. $NM^{-1} < 0$, holds on $\operatorname{Im} P$.*

Proof. From Theorem 4.18 we know that the matrices $Q(t)$ and $\tilde{Q}(t)$ are symmetric solutions of $(\mathcal{R})$ on $[\alpha,\infty)$. Therefore, the quantities $\operatorname{rank}[\tilde{Q}(t)-Q(t)]$ and $\operatorname{ind}[\tilde{Q}(t)-Q(t)]$ are constant on $[\alpha,\infty)$, by Corollary 4.14. According to (2.31) the matrix M is nonsingular and the matrix $M^T N$ is symmetric. Thus, the matrix $M^{T-1}M^T N M^{-1} = NM^{-1}$ is symmetric and

$$\operatorname{rank} M^T N = \operatorname{rank} NM^{-1} = \operatorname{rank} N, \quad \operatorname{ind} M^T N = \operatorname{ind} NM^{-1}. \tag{5.15}$$

The formulas in (5.14) now follow from (5.13) and (5.15). Furthermore, assertions (i) and (ii) are direct consequences of the equalities in (5.14). For the proof of statement (iii) we note that the matrix $\tilde{Q}(t)-Q(t)$ satisfies the inclusion $\operatorname{Im}[\tilde{Q}(t)-Q(t)] \subseteq \operatorname{Im} R_{\mathcal{G}}(t)$ for all $t\in[\alpha,\infty)$ by Theorem 4.18, while the matrix NM^{-1} satisfies the inclusion $\operatorname{Im} NM^{-1} \subseteq \operatorname{Im} P$ by Proposition 2.10. Moreover, the equality $\operatorname{rank} R_{\mathcal{G}}(t) \equiv \operatorname{rank} P =: r$ holds on $[\alpha,\infty)$, by (2.6) and (5.8). Now if $\tilde{Q}(t) > Q(t)$ on $\operatorname{Im} R_{\mathcal{G}}(t)$ for every $t\in[\alpha,\infty)$, then we have that $\operatorname{rank}[\tilde{Q}(t)-Q(t)] = r$ and $\operatorname{ind}[\tilde{Q}(t)-Q(t)] = 0$ on $[\alpha,\infty)$. Consequently, by using (5.14), we obtain the equalities $\operatorname{rank} NM^{-1} = \operatorname{rank} N = r$ and $\operatorname{ind} NM^{-1} = 0$. This means that the matrix NM^{-1} satisfies $NM^{-1} > 0$ on $\operatorname{Im} P$. Conversely, if the inequality $NM^{-1} > 0$ holds on $\operatorname{Im} P$, then $\operatorname{rank} N = \operatorname{rank} NM^{-1} = r$ and $\operatorname{ind} NM^{-1} = 0$. Therefore, $\operatorname{rank}[\tilde{Q}(t)-Q(t)] = r$ and $\operatorname{ind}[\tilde{Q}(t)-Q(t)] = 0$ on $t\in[\alpha,\infty)$, by (5.14). This then shows that $\tilde{Q}(t) > Q(t)$ on $\operatorname{Im} R_{\mathcal{G}}(t)$ for all $t\in[\alpha,\infty)$. Finally, the opposite inequalities can be proven in a similar way. The proof is complete. □

Remark 5.6. As a completion of Corollary 5.5 we note that the matrices $Q(t)$ and $\tilde{Q}(t)$ are equal on the whole interval $[\alpha,\infty)$ if and only if $N = 0$. In this case, according to (2.30), the conjoined bases (X,U) and $(\tilde{X},\tilde{U})$ satisfy $\tilde{X}(t) = X(t)\,M$ and $\tilde{U}(t) = U(t)\,M$ on $[\alpha,\infty)$, and hence on $[a,\infty)$ by the uniqueness of solutions of (H). This result is in full agreement with the last part of Remark 4.23.

6. **Implicit Riccati matrix differential equation.**

In this section we study solution spaces of the implicit Riccati equations (1.6) and

$$R_{\mathcal{G}}(t)\,[Q' + Q\,\mathcal{A}(t) + \mathcal{A}^T(t)\,Q + Q\,\mathcal{B}(t)\,Q - \mathcal{C}(t)]\,R_{\mathcal{G}}(t) = 0 \tag{6.1}$$

on $[\alpha,\infty) \subseteq [\alpha_\infty,\infty)$. These implicit Riccati equations were used in [13, Section 6] in several criteria characterizing the nonnegativity and positivity of the associated quadratic functional. The main contributions of this section (Theorem 6.3 and Corollary 6.4) show that under certain assumption we can transfer the problem of solving the implicit Riccati equations (6.1) and (1.6) into a problem of solving the explicit Riccati equation $(\mathcal{R})$.

In the first result we prove that the two implicit Riccati equations (6.1) and (1.6) are equivalent in terms of their solutions spaces.

Lemma 6.1. *Let $\mathcal{G}$ be a genus of conjoined bases of* (H) *and let $R_{\mathcal{G}}(t)$ be the orthogonal projector in* (3.11). *The sets of solutions of equations* (1.6) *and* (6.1) *coincide, i.e., a matrix $Q(t)$ solves* (1.6) *on a subinterval $[\alpha,\infty) \subseteq [\alpha_\infty,\infty)$ if and only if $Q(t)$ solves* (6.1) *on $[\alpha,\infty)$.*

Proof. Let $\mathcal{G}$ and $R_{\mathcal{G}}(t)$ be as in the lemma and fix $\alpha\in[\alpha_\infty,\infty]$. Moreover, let $Q(t)$ be an $n\times n$ piecewise continuously differentiable matrix-valued function on $[\alpha,\infty)$

and define the functions (we omit the argument t)

$$\left.\begin{aligned} \mathcal{E}_1 &:= R_{\mathcal{G}}\,[Q' + QA + A^TQ + QBQ - C]\,R_{\mathcal{G}}, \\ \mathcal{E}_2 &:= R_{\mathcal{G}}\,[Q' + Q\mathcal{A} + \mathcal{A}^TQ + Q\mathcal{B}Q - \mathcal{C}]\,R_{\mathcal{G}}, \end{aligned}\right\} \tag{6.2}$$

on $[\alpha,\infty)$. By using (1.3) and (4.3) together with the identity $[R_{\mathcal{G}}(t)]^2 = R_{\mathcal{G}}(t)$ for every $t \in [\alpha,\infty)$ we then obtain (suppressing the argument t)

$$\begin{aligned} \mathcal{E}_1 &\overset{(6.2)}{=} R_{\mathcal{G}}\,Q'R_{\mathcal{G}} + R_{\mathcal{G}}\,Q\,AR_{\mathcal{G}} + R_{\mathcal{G}}A^TQ\,R_{\mathcal{G}} + R_{\mathcal{G}}\,QB\,QR_{\mathcal{G}} - R_{\mathcal{G}}CR_{\mathcal{G}} \\ &\overset{(1.3),\,(4.3)}{=} R_{\mathcal{G}}\,Q'R_{\mathcal{G}} + R_{\mathcal{G}}\,Q\,\mathcal{A}R_{\mathcal{G}} + R_{\mathcal{G}}\mathcal{A}^TQ\,R_{\mathcal{G}} + R_{\mathcal{G}}\,Q\,\mathcal{B}\,QR_{\mathcal{G}} - R_{\mathcal{G}}\,\mathcal{C}R_{\mathcal{G}} \overset{(6.2)}{=} \mathcal{E}_2 \end{aligned}$$

on $[\alpha,\infty)$, which proves directly the statement of the lemma. $\square$

Remark 6.2. It is easy to see that for a given orthogonal projector $R_{\mathcal{G}}(t)$ in (3.11) any matrix $Q(t)$, which solves the Riccati equation ($\mathcal{R}$) on some subinterval $[\alpha,\infty) \subseteq [\alpha_\infty,\infty)$, satisfies also the implicit Riccati equation (6.1) on $[\alpha,\infty)$.

Following the above remark, we now establish the opposite relation between the solutions of the implicit Riccati equation (6.1) and the Riccati equation ($\mathcal{R}$).

Theorem 6.3. *Let $\mathcal{G}$ be a genus of conjoined bases of* (H) *with the orthogonal projector $R_{\mathcal{G}}(t)$ in* (3.11). *Let $Q(t)$ be a solution of the implicit Riccati equation* (6.1) *on $[\alpha,\infty) \subseteq [\alpha_\infty,\infty)$. Then the matrix $R_{\mathcal{G}}(t)\,Q(t)\,R_{\mathcal{G}}(t)$ solves ($\mathcal{R}$) on $[\alpha,\infty)$.*

Proof. Let $R_{\mathcal{G}}(t)$ and $Q(t)$ be as in the theorem. With the aid of (4.1), (4.2), (6.1), and the equalities (suppressing the argument t) $R_{\mathcal{G}}\,\mathcal{C}R_{\mathcal{G}} = \mathcal{C}$ and $\mathcal{B} = R_{\mathcal{G}}\,\mathcal{B}R_{\mathcal{G}}$ on $[\alpha,\infty)$ we get

$$\begin{aligned} (R_{\mathcal{G}}\,QR_{\mathcal{G}})' &= R_{\mathcal{G}}'\,QR_{\mathcal{G}} + R_{\mathcal{G}}\,Q'R_{\mathcal{G}} + R_{\mathcal{G}}\,QR_{\mathcal{G}}' \\ &\overset{(4.1),\,(4.2)}{=} [R_{\mathcal{G}},\mathcal{A}^T]\,QR_{\mathcal{G}} + R_{\mathcal{G}}\,Q'R_{\mathcal{G}} + R_{\mathcal{G}}\,Q\,[\mathcal{A},R_{\mathcal{G}}] \\ &\overset{(6.1)}{=} \mathcal{C} - (R_{\mathcal{G}}\,QR_{\mathcal{G}})\,\mathcal{A} - \mathcal{A}^T(R_{\mathcal{G}}\,QR_{\mathcal{G}}) - (R_{\mathcal{G}}\,QR_{\mathcal{G}})\,\mathcal{B}\,(R_{\mathcal{G}}\,QR_{\mathcal{G}}) \end{aligned}$$

on $[\alpha,\infty)$. Hence, the matrix $R_{\mathcal{G}}(t)\,Q(t)\,R_{\mathcal{G}}(t)$ solves ($\mathcal{R}$) on $[\alpha,\infty)$. $\square$

The results in Theorem 6.3 and Lemma 6.1 yield the following.

Corollary 6.4. *Let $\mathcal{G}$ be a genus of conjoined bases of* (H) *and let $R_{\mathcal{G}}(t)$ be the corresponding orthogonal projector in* (3.11). *Moreover, let $Q(t)$ be a symmetric matrix defined on $[\alpha,\infty) \subseteq [\alpha_\infty,\infty)$ such that condition* (1.4) *holds. Then the following statements are equivalent.*

(i) *The matrix $Q(t)$ solves the Riccati equation ($\mathcal{R}$) on $[\alpha,\infty)$.*
(ii) *The matrix $Q(t)$ solves the implicit Riccati equation* (6.1) *on $[\alpha,\infty)$.*
(iii) *The matrix $Q(t)$ solves the implicit Riccati equation* (1.6) *on $[\alpha,\infty)$.*

Proof. The implication (i) $\Rightarrow$ (ii) follows by Remark 6.2. The equivalence of the assertions in (ii) and (iii) is a direct consequence of Lemma 6.1. Now assume (ii), i.e., suppose that the matrix $Q(t)$ is a solution of (6.1) on $[\alpha,\infty)$. The result of Theorem 6.3 and the identities

$$R_{\mathcal{G}}(t)\,Q(t)\,R_{\mathcal{G}}(t) \overset{(1.4)}{=} Q(t)\,R_{\mathcal{G}}(t) = Q^T(t)\,R_{\mathcal{G}}(t) = [R_{\mathcal{G}}(t)\,Q(t)]^T \overset{(1.4)}{=} Q(t)$$

for $t \in [\alpha,\infty)$ then imply that $Q(t)$ solves ($\mathcal{R}$) on $[\alpha,\infty)$, showing (i). $\square$

7. **Distinguished solutions at infinity.** In this section we study, for a given genus $\mathcal{G}$, symmetric solutions of the Riccati equation ($\mathcal{R}$), which correspond to principal solutions of (H) at infinity belonging to the genus $\mathcal{G}$. This correspondence is based on the results in Theorems 4.18 and 4.21 and in Remark 4.20. We introduce the notion of a distinguished solution of ($\mathcal{R}$) at infinity (Definition 7.1) and prove its main properties. In particular, we establish the results about distinguished solutions of ($\mathcal{R}$) at infinity regarding their relationship to principal solutions at infinity (Theorems 7.4 and 7.5) and to the nonoscillation of system (H) at infinity (Theorem 7.8), their interval of existence (Theorem 7.13), their mutual classification within the genus $\mathcal{G}$ (Theorem 7.15), and their minimality in a suitable sense (Theorems 7.16 and 7.18).

It may be surprising that these results comply with the known theory of distinguished solutions of the Riccati equation (R) for a controllable system (H) only partially. In many aspects the presented theory for general uncontrollable system (H) is substantially different. This is related to the nature of the problem, since for each genus $\mathcal{G}$ of conjoined bases of (H) there is a different Riccati equation ($\mathcal{R}$), but even within one genus $\mathcal{G}$ there may be many distinguished solutions of ($\mathcal{R}$) at infinity. We discuss these issues in Remark 7.25 at the end of this section. We note that the true uniqueness and minimality of the distinguished solution of ($\mathcal{R}$) at infinity is satisfied only in the minimal genus $\mathcal{G}_{\min}$ (see Theorem 7.23).

The following definition extends the notion of a distinguished solution (also called a principal solution) of (R) at infinity for a controllable system (H) in [7, pg. 53].

Definition 7.1. Let $\mathcal{G}$ be a genus of conjoined bases of (H) with the orthogonal projector $R_{\mathcal{G}}(t)$ in (3.11). A symmetric solution $\hat{Q}(t)$ of the Riccati equation ($\mathcal{R}$) is said to be a *distinguished solution at infinity* if the matrix $\hat{Q}(t)$ is defined on an interval $[\alpha,\infty)\subseteq[\alpha_\infty,\infty)$ and its corresponding matrix $\hat{F}_\alpha(t)$ in (4.18) satisfies $\hat{F}_\alpha^\dagger(t)\to 0$ as $t\to\infty$.

The notion in Definition 7.1 also extends the distinguished solution of (R) introduced by W. T. Reid in [21, Section IV] and [23, Section 2.7], which in our context corresponds to the maximal genus $\mathcal{G}=\mathcal{G}_{\max}$ (for which $R_{\mathcal{G}}(t)\equiv I$).

Remark 7.2. When it is clear from the context, we will often drop the term "at infinity" in the terminology in Definition 7.1. We also remark that a distinguished solution of the Riccati equation ($\mathcal{R}$) associated with the genus $\mathcal{G}$ is also defined by the property $\hat{D}_\alpha=0$ with the matrix $\hat{D}_\alpha$ in (4.19) corresponding to $\hat{F}_\alpha(t)$.

In the next auxiliary statement we show that the property of being a distinguished solution of ($\mathcal{R}$) is invariant under the multiplication by the orthogonal projector $R_{\mathcal{G}}(t)$. This property will be utilized in the proofs of the subsequent main results.

Lemma 7.3. *Let $\mathcal{G}$ be a genus of conjoined bases of* (H) *with the matrix $R_{\mathcal{G}}(t)$ in* (3.11)*. Let $Q(t)$ be a symmetric solution of the Riccati equation* ($\mathcal{R}$) *on the interval $[\alpha,\infty)\subseteq[\alpha_\infty,\infty)$. Then $Q(t)$ is a distinguished solution of* ($\mathcal{R}$) *at infinity with respect to $[\alpha,\infty)$ if and only if the matrix $R_{\mathcal{G}}(t)\,Q(t)\,R_{\mathcal{G}}(t)$ is a distinguished solution of* ($\mathcal{R}$) *at infinity with respect to $[\alpha,\infty)$.*

Proof. From Theorem 4.3 we know that the matrix $R_{\mathcal{G}}(t)\,Q(t)\,R_{\mathcal{G}}(t)$ solves ($\mathcal{R}$) on $[\alpha,\infty)$. And since by Remark 4.11(ii) the matrices $Q(t)$ and $R_{\mathcal{G}}(t)\,Q(t)\,R_{\mathcal{G}}(t)$ have the same F-matrices in (4.18) with respect to the interval $[\alpha,\infty)$, the statement follows directly from Definition 7.1. $\square$

The following two results show that in the context of Theorems 4.18 and 4.21 the distinguished solutions of ($\mathcal{R}$) correspond to the principal solutions of (H) at infinity from the genus $\mathcal{G}$.

Theorem 7.4. *Let $\mathcal{G}$ be a genus of conjoined bases of (H) and $R_{\mathcal{G}}(t)$ be the projector in (3.11). Moreover, let $\hat{Q}(t)$ be a distinguished solution of ($\mathcal{R}$) at infinity with respect to the interval $[\alpha,\infty)\subseteq[\alpha_\infty,\infty)$. Then every conjoined basis $(\hat{X},\hat{U})$ of (H), which is associated with $\hat{Q}(t)$ on $[\alpha,\infty)$ via Theorem 4.21, is a principal solution of (H) at infinity with respect to $[\alpha,\infty)$ belonging to the genus $\mathcal{G}$.*

Proof. Let $R_{\mathcal{G}}(t)$ and $\hat{Q}(t)$ be as in the theorem. According to Remark 7.2 the matrix $\hat{D}_\alpha$ in (4.19) corresponding to $\hat{Q}(t)$ satisfies $\hat{D}_\alpha=0$. Let $(\hat{X},\hat{U})$ be a conjoined basis of (H), which is associated with the matrix $\hat{Q}(t)$ on $[\alpha,\infty)$ via Theorem 4.21. Then $(\hat{X},\hat{U})$ belongs to the genus $\mathcal{G}$ such that $(\hat{X},\hat{U})$ has constant kernel on $[\alpha,\infty)$. Moreover, if $\hat{T}_\alpha$ is the T-matrix in (2.13) associated with $(\hat{X},\hat{U})$ through the matrix $\hat{S}_\alpha$ in (2.11), then we have $\operatorname{rank}\hat{T}_\alpha=\operatorname{rank}\hat{D}_\alpha=0$, by Remark 5.2(ii). Hence, $\hat{T}_\alpha=0$ and $(\hat{X},\hat{U})$ is a principal solution at infinity. ☐

Theorem 7.5. *Let $(\hat{X},\hat{U})$ be a principal solution of (H) at infinity with respect to the interval $[\alpha,\infty)$, which belongs to a genus $\mathcal{G}$. Moreover, let $\hat{Q}(t)$ be the Riccati quotient in (2.7) associated with $(\hat{X},\hat{U})$ on $[\alpha,\infty)$. Then $\hat{Q}(t)$ is a distinguished solution of the Riccati equation ($\mathcal{R}$) at infinity with respect to $[\alpha,\infty)$.*

Proof. By using Proposition 3.2(i) we have the equality $d[\alpha,\infty)=d_\infty$, which means that $[\alpha,\infty)\subseteq[\alpha_\infty,\infty)$, by (2.19). Moreover, the matrix $\hat{T}_\alpha$ in (2.13) associated with $(\hat{X},\hat{U})$ satisfies $\hat{T}_\alpha=0$. From Theorem 4.18 it follows that the matrix $\hat{Q}(t)$ is a symmetric solution of the Riccati equation ($\mathcal{R}$) on $[\alpha,\infty)$. Finally, if $\hat{D}_\alpha$ is the matrix in (4.19), which corresponds to $\hat{Q}$ through its F-matrix $\hat{F}_\alpha(t)$ in (4.18), then $\operatorname{rank}\hat{D}_\alpha=\operatorname{rank}\hat{T}_\alpha=0$, by Remark 5.2(ii). Thus, $\hat{D}_\alpha=0$ and $\hat{Q}(t)$ is a distinguished solution at infinity, by Remark 7.2. ☐

Remark 7.6. We note that according to Theorem 4.18 the distinguished solution $\hat{Q}(t)$ at infinity in Theorem 7.5 satisfies the additional property (1.4), i.e., the inclusion $\operatorname{Im}\hat{Q}(t)\subseteq R_{\mathcal{G}}(t)$ for all $t\in[\alpha,\infty)$. In particular, the latter relation together with the symmetry of the matrix $\hat{Q}(t)$ on $[\alpha,\infty)$ yields the identity $\hat{Q}(t)=R_{\mathcal{G}}(t)\,\hat{Q}(t)\,R_{\mathcal{G}}(t)$ for every $t\in[\alpha,\infty)$. Moreover, from Lemma 7.3 it follows that every symmetric solution $Q(t)$ of ($\mathcal{R}$) on $[\alpha,\infty)$, for which the matrix $R_{\mathcal{G}}(t)\,Q(t)\,R_{\mathcal{G}}(t)$ is the Riccati quotient in (2.7) associated with $(\hat{X},\hat{U})$, is also a distinguished solution at infinity with respect to $[\alpha,\infty)$. In general, however, such a matrix $Q(t)$ does not need to satisfy the inclusion in (1.4).

From Theorems 7.4 and 7.5 it follows that the property of the existence of a principal solution of (H) at infinity in the genus $\mathcal{G}$, as stated in [29, Theorem 7.12], transfers naturally to the existence of a distinguished solution at infinity of the associated Riccati equation ($\mathcal{R}$).

Corollary 7.7. *Let $\mathcal{G}$ be a genus of conjoined bases of (H) with the orthogonal projector $R_{\mathcal{G}}(t)$ in (3.11). Then there exists a principal solution of (H) at infinity belonging to the genus $\mathcal{G}$ if and only if there exists a distinguished solution of the Riccati equation ($\mathcal{R}$) at infinity. In this case, the set of all Riccati quotients in (2.7), which correspond to the principal solutions $(\hat{X},\hat{U})$ of (H) at infinity from the* *genus $\mathcal{G}$, coincides with the set of all matrices $R_{\mathcal{G}}\,\hat{Q}R_{\mathcal{G}}$, where $\hat{Q}$ is a distinguished solution of* ($\mathcal{R}$) *at infinity.*

Proof. The statement follows directly from Theorems 7.4 and 7.5 and from Remark 7.6. □

In the following result we characterize the nonoscillation of system (H) in terms of the existence of a distinguished solution of the Riccati equation ($\mathcal{R}$) in a given (or every) genus $\mathcal{G}$. This corresponds to [29, Theorems 7.6 and 7.12] regarding the principal solutions of (H) at infinity.

Theorem 7.8. *Assume* (1.1). *Then the following statements are equivalent.*

(i) *System* (H) *is nonoscillatory.*
(ii) *There exists a distinguished solution of equation* ($\mathcal{R}$) *for some genus $\mathcal{G}$.*
(iii) *There exists a distinguished solution of equation* ($\mathcal{R}$) *for every genus $\mathcal{G}$.*

The proof of Theorem 7.8 is displayed below after the following two remarks.

Remark 7.9. The result in Theorem 7.8 justifies the development of the theory of genera of conjoined bases for possibly oscillatory system (H). Of course, assuming that system (H) is nonoscillatory, then it is sufficient to use the theory of genera of conjoined bases from [29, Section 6] and [31, Section 4] for the construction of distinguished solutions of the Riccati equation ($\mathcal{R}$) for a genus $\mathcal{G}$. It is the converse to this implication, which requires a more general approach, since in this case we need to define the coefficients of equation ($\mathcal{R}$) without the assumption of nonoscillation of system (H). This natural requirement was the initial motivation for the study presented in [27].

Remark 7.10. We note that the result in Theorem 7.8 remains valid also with the additional condition (1.4) for solutions $Q(t)$ of ($\mathcal{R}$) in parts (ii) and (iii). More precisely, system (H) is nonoscillatory if and only if there exists a distinguished solution of ($\mathcal{R}$) at infinity for some (and hence for every) genus $\mathcal{G}$, which satisfies condition (1.4) for all sufficiently large $t \in [\alpha_\infty, \infty)$. This observation follows directly from Lemma 7.3 and Theorem 7.8.

Proof of Theorem 7.8. If (H) is nonoscillatory, then by Remark 3.13 for any genus $\mathcal{G}$ of conjoined bases of (H) there exists a principal solution of (H) at infinity belonging to $\mathcal{G}$. In turn, there exists a distinguished solution of the Riccati equation ($\mathcal{R}$) at infinity for every genus $\mathcal{G}$, by Corollary 7.7. Moreover, assertion (iii) implies (ii) trivially. Finally, by using Corollary 7.7 once more, assertion (ii), that is, the existence of a distinguished solution of ($\mathcal{R}$) at infinity for some genus $\mathcal{G}$, means that there exists a principal solution of (H) at infinity, which belongs to $\mathcal{G}$. Since every principal solution is a nonoscillatory conjoined basis, system (H) is nonoscillatory, by Proposition 2.1. This shows the validity of (i) and completes the proof. □

The next two results deal with the interval of existence of distinguished solutions of ($\mathcal{R}$). In particular, we determine the maximal interval of existence for each particular distinguished solution of ($\mathcal{R}$). Moreover, we show that this maximal interval is the same for all distinguished solutions of ($\mathcal{R}$) as well as for all genera $\mathcal{G}$.

Theorem 7.11. *Assume* (1.1) *and let $\mathcal{G}$ be a genus of conjoined bases of* (H) *with the matrix $R_{\mathcal{G}}(t)$ in* (3.11). *Moreover, let $\hat{Q}(t)$ be a distinguished solution of the Riccati equation* ($\mathcal{R}$) *at infinity with respect to the interval $[\alpha, \infty) \subseteq [\alpha_\infty, \infty)$.*

Then the matrix $\hat{Q}(t)$ is a distinguished solution of ($\mathcal{R}$) *at infinity also with respect to the interval $[\beta,\infty)$ for every $\beta \geq \alpha$.*

Proof. Let $(\hat{X},\hat{U})$ be a conjoined basis of (H) corresponding to $\hat{Q}(t)$ on $[\alpha,\infty)$ via Theorem 4.21. In particular, the matrix $R_{\mathcal{G}}(t)\,\hat{Q}(t)\,R_{\mathcal{G}}(t)$ is the Riccati quotient in (2.7) associated with $(\hat{X},\hat{U})$ on $[\alpha,\infty)$. Moreover, from Theorem 7.4 we know that $(\hat{X},\hat{U})$ is a principal solution of (H) at infinity with respect to $[\alpha,\infty)$, which belongs to the genus $\mathcal{G}$. Fix now $\beta \geq \alpha$. Then $(\hat{X},\hat{U})$ is a principal solution of (H) at infinity with respect to $[\beta,\infty)$, by Proposition 3.2(i). Consequently, the matrix $R_{\mathcal{G}}(t)\,\hat{Q}(t)\,R_{\mathcal{G}}(t)$ is a distinguished solution of ($\mathcal{R}$) at infinity with respect to $[\beta,\infty)$, by Theorem 7.5. Finally, by using Lemma 7.3 we conclude that also the matrix $\hat{Q}(t)$ is a distinguished solution of ($\mathcal{R}$) at infinity with respect to the interval $[\beta,\infty)$. □

Remark 7.12. For a given distinguished solution $\hat{Q}(t)$ of the Riccati equation ($\mathcal{R}$) at infinity we define the point $\alpha_{\hat{Q}} \in [\alpha_\infty,\infty)$ by

$$\alpha_{\hat{Q}} := \inf\big\{\alpha \in [a,\infty),\ \hat{Q}(t) \text{ is a distinguished solution of } (\mathcal{R}) \text{ with respect to } [\alpha,\infty)\big\}. \tag{7.1}$$

The result in Theorem 7.11 then implies that $\hat{Q}(t)$ is a distinguished solution of ($\mathcal{R}$) with respect to $[\alpha,\infty)$ for every $\alpha \in (\alpha_{\hat{Q}},\infty)$. In fact, the set $(\alpha_{\hat{Q}},\infty)$ is the maximal open interval on which the matrix $\hat{Q}(t)$ exists as a solution of ($\mathcal{R}$). Indeed, if the matrix $\hat{Q}(t)$ solves the equation ($\mathcal{R}$) on the interval $[\alpha,\infty) \subseteq [\alpha_\infty,\infty)$, then according to Remark 5.2(ii) the corresponding matrix D_α in (4.19) satisfies $D_\alpha = 0$, because $d[\alpha,\infty) = d_\infty$, by (2.19). Thus, $\hat{Q}(t)$ is a distinguished solution of ($\mathcal{R}$) with respect to the interval $[\alpha,\infty)$, by Remark 7.2.

Theorem 7.13. *Assume that* (1.1) *holds and system* (H) *is nonoscillatory with $\hat{\alpha}_{\min}$ defined in* (3.1)*. Let $\mathcal{G}$ be a genus of conjoined bases of* (H) *and let $R_{\mathcal{G}}(t)$ be its corresponding orthogonal projector in* (3.11)*. Moreover, let $\hat{Q}(t)$ be a distinguished solution of the Riccati equation* ($\mathcal{R}$) *at infinity with $\alpha_{\hat{Q}}$ defined in* (7.1)*. Then the equality $\alpha_{\hat{Q}} = \hat{\alpha}_{\min}$ holds.*

Proof. Let $\alpha \in [\alpha_\infty,\infty)$ be such that the matrix $\hat{Q}(t)$ is a distinguished solution of ($\mathcal{R}$) with respect to the interval $[\alpha,\infty)$. Let $(\hat{X},\hat{U})$ be a conjoined basis of (H) at infinity with respect to $[\alpha,\infty)$, which is associated with $\hat{Q}(t)$ via Theorem 4.21. Then $(\hat{X},\hat{U})$ is a principal solution of (H) at infinity with respect to the interval $[\alpha,\infty)$, by Theorem 7.4. Moreover, from Theorem 3.4 we know that $(\hat{X},\hat{U})$ is a principal solution with respect to the maximal open interval $(\hat{\alpha}_{\min},\infty)$. Thus, we have the inequality $\hat{\alpha}_{\min} \leq \alpha$. And since $\alpha \in [\alpha_\infty,\infty)$ was chosen arbitrarily with regard to $\hat{Q}(t)$, we obtain that $\alpha_{\hat{Q}} \leq \hat{\alpha}_{\min}$, by (7.1). Now we show that the last inequality is implemented as the equality. Suppose that $\alpha_{\hat{Q}} < \hat{\alpha}_{\min}$. According to (7.1) there exists $\beta \in (\alpha_{\hat{Q}},\hat{\alpha}_{\min})$ such that $\hat{Q}(t)$ is a distinguished solution of ($\mathcal{R}$) at infinity with respect to the interval $[\beta,\infty)$. In turn, the conjoined basis (X,U) in Theorem 4.21 applied to $Q(t) := \hat{Q}(t)$ on $[\beta,\infty)$ is a principal solution of (H) at infinity with respect to $[\beta,\infty)$, by Theorem 7.4. Applying formula (3.3) and Theorem 3.4 with $(\hat{X},\hat{U}) := (X,U)$ then yields the inequality $\beta \geq \hat{\alpha}_{\min}$, which is a contradiction. Therefore, $\alpha_{\hat{Q}} = \hat{\alpha}_{\min}$ holds and the proof is complete. □

Remark 7.14. Given a genus $\mathcal{G}$ of conjoined bases of (H) with the matrix $R_{\mathcal{G}}(t)$ in (3.11), from Theorem 7.13 it follows that any distinguished solution $\hat{Q}(t)$ of ($\mathcal{R}$) is defined on the maximal interval $(\hat{\alpha}_{\min}, \infty)$ and the corresponding matrix $F_\alpha(t)$ in (4.18) satisfies $\hat{F}^\dagger_\alpha(t) \to 0$ as $t \to \infty$ for every $\alpha > \hat{\alpha}_{\min}$.

In the following result we present a mutual classification of all distinguished solutions of the Riccati equation ($\mathcal{R}$). This classification is formulated in terms of the initial values of the involved distinguished solutions at some point α from the maximal interval $(\hat{\alpha}_{\min}, \infty)$.

Theorem 7.15. *Assume that* (1.1) *holds and system* (H) *is nonoscillatory with* $\hat{\alpha}_{\min}$ *and* $R_{\Lambda\infty}(t)$ *defined in* (3.1) *and* (3.7)*, respectively. Let* $\mathcal{G}$ *be a genus of conjoined bases of* (H) *and let* $R_{\mathcal{G}}(t)$ *be the matrix in* (3.11)*. Moreover, let* $\hat{Q}(t)$ *be a distinguished solution of the Riccati equation* ($\mathcal{R}$) *at infinity. Then a symmetric solution* $Q(t)$ *of* ($\mathcal{R}$) *defined on a neighborhood of some point* $\alpha \in (\hat{\alpha}_{\min}, \infty)$ *is a distinguished solution at infinity if and only if*

$$R_{\Lambda\infty}(\alpha)\, Q(\alpha)\, R_{\Lambda\infty}(\alpha) = R_{\Lambda\infty}(\alpha)\, \hat{Q}(\alpha)\, R_{\Lambda\infty}(\alpha). \tag{7.2}$$

Proof. Fix $\alpha \in (\hat{\alpha}_{\min}, \infty)$ and let $\hat{Q}(t)$ be as in the theorem. From Definition 7.1 and Remark 7.14 we know that $\hat{Q}(t)$ is a distinguished solution of ($\mathcal{R}$) at infinity with respect to the interval $[\alpha, \infty)$. Moreover, let $(\hat{X}, \hat{U})$ be a conjoined basis of (H), which corresponds to $\hat{Q}(t)$ on $[\alpha, \infty)$ via Theorem 4.21. Then $(\hat{X}, \hat{U})$ is a principal solution of (H) at infinity with respect to $[\alpha, \infty)$ belonging to the genus $\mathcal{G}$, by Theorem 7.4. Let $\hat{S}_\alpha(t)$ be the S-matrix in (2.11) associated with $(\hat{X}, \hat{U})$. Suppose that $Q(t)$ is a distinguished solution of ($\mathcal{R}$) at infinity. Thus, $Q(t)$ is a distinguished solution of ($\mathcal{R}$) with respect to $[\alpha, \infty)$ and the corresponding conjoined basis (X, U) of (H) in Theorem 4.21 is a principal solution with respect to $[\alpha, \infty)$, which belongs to the genus $\mathcal{G}$. From Proposition 3.14 it then follows that there exist matrices $\hat{M}, \hat{N} \in \mathbb{R}^{n\times n}$ such that

$$X(\alpha) = \hat{X}(\alpha)\, \hat{M}, \quad U(\alpha) = \hat{U}(\alpha)\, \hat{M} + \hat{X}^{\dagger T}(\alpha)\, \hat{N}, \tag{7.3}$$

$$\hat{M} \text{ is nonsingular}, \;\; \hat{M}^T \hat{N} = \hat{N}^T \hat{M}, \;\; \operatorname{Im} \hat{N} \subseteq \operatorname{Im} \hat{P}, \;\; P_{\hat{\mathcal{S}}\infty}\, \hat{N} \hat{M}^{-1} P_{\hat{\mathcal{S}}\infty} = 0, \tag{7.4}$$

where $\hat{P}$ and $P_{\hat{\mathcal{S}}\infty}$ are the matrices in (2.5), (2.12), and (3.2) associated with the functions $\hat{X}(t)$ and $\hat{S}_\alpha$ on $(\hat{\alpha}_{\min}, \infty)$, respectively. In particular, the matrices $\hat{M}$ and $\hat{N}$ in (7.3)–(7.4) represent the conjoined basis (X, U) in terms of $(\hat{X}, \hat{U})$ on $[\alpha, \infty)$ in Proposition 2.10 and Remark 2.11. Moreover, the matrices $R_{\mathcal{G}}(t)\, Q(t)\, R_{\mathcal{G}}(t)$ and $R_{\mathcal{G}}(t)\, \hat{Q}(t)\, R_{\mathcal{G}}(t)$ are the Riccati quotients in (2.7) associated with (X, U) and $(\hat{X}, \hat{U})$ on $[\alpha, \infty)$, by Theorem 4.21(ii). Consequently, according to (5.7) in Theorem 5.3 with $(X, U) := (\hat{X}, \hat{U})$, $(\tilde{X}, \tilde{U}) := (X, U)$, $Q := R_{\mathcal{G}}\, \hat{Q} R_{\mathcal{G}}$, $\tilde{Q} := R_{\mathcal{G}}\, Q R_{\mathcal{G}}$, $S_\alpha := \hat{S}_\alpha$, $M := \hat{M}$, $N := \hat{N}$, and by using (2.36) and $\hat{S}_\alpha(\alpha) = 0$ we obtain the identity

$$R_{\mathcal{G}}(\alpha)\, Q(\alpha)\, R_{\mathcal{G}}(\alpha) = R_{\mathcal{G}}(\alpha)\, \hat{Q}(\alpha)\, R_{\mathcal{G}}(\alpha) + \hat{X}^{\dagger T}(\alpha)\, \hat{N} \hat{M}^{-1} \hat{X}^\dagger(\alpha). \tag{7.5}$$

In order to simplify the notation we set

$$Z := R_{\Lambda\infty}(\alpha)\, Q(\alpha)\, R_{\Lambda\infty}(\alpha), \quad \hat{Z} := R_{\Lambda\infty}(\alpha)\, \hat{Q}(\alpha)\, R_{\Lambda\infty}(\alpha). \tag{7.6}$$

Since we have $R_{\Lambda\infty}(\alpha)\, R_{\mathcal{G}}(\alpha) = R_{\Lambda\infty}(\alpha) = R_{\mathcal{G}}(\alpha)\, R_{\Lambda\infty}(\alpha)$ and $\hat{X}^{\dagger}(\alpha)\, R_{\Lambda\infty}(\alpha) = P_{\hat{\mathcal{S}}\infty}\, \hat{X}^{\dagger}(\alpha)\, R_{\Lambda\infty}(\alpha)$ by (3.10), formula (7.5) and the last condition in (7.4) imply

$$\begin{aligned} Z &\overset{(7.6)}{=} R_{\Lambda\infty}(\alpha)\, Q(\alpha)\, R_{\Lambda\infty}(\alpha) = R_{\Lambda\infty}(\alpha)\, R_{\mathcal{G}}(\alpha)\, Q(\alpha)\, R_{\mathcal{G}}(\alpha)\, R_{\Lambda\infty}(\alpha) \\ &\overset{(7.5)}{=} R_{\Lambda\infty}(\alpha) \left[R_{\mathcal{G}}(\alpha)\, \hat{Q}(\alpha)\, R_{\mathcal{G}}(\alpha) + \hat{X}^{\dagger T}(\alpha)\, \hat{N}\hat{M}^{-1}\hat{X}^{\dagger}(\alpha) \right] R_{\Lambda\infty}(\alpha) \\ &= R_{\Lambda\infty}(\alpha)\, \hat{Q}(\alpha)\, R_{\Lambda\infty}(\alpha) + R_{\Lambda\infty}(\alpha)\, \hat{X}^{\dagger T}(\alpha)\, \hat{N}\hat{M}^{-1}\hat{X}^{\dagger}(\alpha)\, R_{\Lambda\infty}(\alpha) \\ &\overset{(7.6)}{=} \hat{Z} + R_{\Lambda\infty}(\alpha)\, \hat{X}^{\dagger T}(\alpha)\, P_{\hat{\mathcal{S}}\infty}\, \hat{N}\hat{M}^{-1} P_{\hat{\mathcal{S}}\infty}\, \hat{X}^{\dagger}(\alpha)\, R_{\Lambda\infty}(\alpha) = \hat{Z}. \end{aligned}$$

Thus, with respect to (7.6) the matrix $Q(\alpha)$ satisfies equality (7.2). Conversely, assume that $Q(t)$ is a symmetric solution of ($\mathcal{R}$) defined on a neighborhood of α such that the condition in (7.2) holds. We will use the notation in (7.6). Let $\hat{F}_\alpha(t)$ be the F-matrix corresponding to $\hat{Q}(t)$ in (4.18) and set $\hat{G} := Q(\alpha) - \hat{Q}(\alpha)$. Then we have $R_{\Lambda\infty}(\alpha)\, \hat{G} R_{\Lambda\infty}(\alpha) = 0$, by (7.2). Next we will show that the matrix $I + \hat{F}_\alpha(t)\, \hat{G}$ is nonsingular on $[\alpha, \infty)$. Fix $t \in [\alpha, \infty)$ and let $v \in \mathbb{R}^n$ be such that $[I + \hat{F}_\alpha(t)\, \hat{G}]\, v = 0$, that is, $v = -\hat{F}_\alpha(t)\, \hat{G} v$. The last equality together with (5.1) imply that $v \in \operatorname{Im} \hat{F}_\alpha(t) \subseteq \operatorname{Im} R_{\Lambda\infty}(\alpha)$. Hence, $v = R_{\Lambda\infty}(\alpha)\, v$ and since $\hat{F}_\alpha(t) = R_{\Lambda\infty}(\alpha)\, \hat{F}_\alpha(t) = \hat{F}_\alpha(t)\, R_{\Lambda\infty}(\alpha)$ by the symmetry of $\hat{F}_\alpha(t)$ and $R_{\Lambda\infty}(\alpha)$, we get $v = -\hat{F}_\alpha(t)\, \hat{G} v = -\hat{F}_\alpha(t)\, R_{\Lambda\infty}(\alpha)\, \hat{G} R_{\Lambda\infty}(\alpha)\, v = 0$. Therefore, the matrix $I + \hat{F}_\alpha(t)\, \hat{G}$ is nonsingular for every $t \in [\alpha, \infty)$. Consequently, according to Remark 4.13 with $Q := \hat{Q}$, $\tilde{Q} := Q$, and $G := \hat{G}$, we conclude that the symmetric matrix $Q(t)$ solve equation ($\mathcal{R}$) on the whole interval $[\alpha, \infty)$. Let (X, U) be a conjoined basis of (H) associated with $Q(t)$ on $[\alpha, \infty)$ via Theorem 4.21. Then (X, U) has constant kernel on $[\alpha, \infty)$ and belongs to the genus $\mathcal{G}$. Therefore, the identities $\operatorname{Im} X(t) = \operatorname{Im} R_{\mathcal{G}}(t) = \operatorname{Im} \hat{X}(t)$ hold for all $t \in [\alpha, \infty)$, by Remark 3.13. Hence, from Proposition 2.10 and Remark 2.11(i) with $(X, U) := (\hat{X}, \hat{U})$ and $(\tilde{X}, \tilde{U}) := (X, U)$ it then follows that there exists matrices $\hat{M}, \hat{N} \in \mathbb{R}^{n \times n}$ such that the formulas in (7.3) and the first three conditions in (7.4) hold. Similarly as in the first part of the proof, the matrix $R_{\mathcal{G}}(t)\, Q(t)\, R_{\mathcal{G}}(t)$ is the Riccati quotient in (2.7) associated with (X, U) on $[\alpha, \infty)$ and the equality in (7.5) holds. Moreover, multiplying (7.5) by the matrix $R_{\Lambda\infty}(\alpha)$ from the both sides and using the identities $R_{\Lambda\infty}(\alpha)\, R_{\mathcal{G}}(\alpha) = R_{\Lambda\infty}(\alpha) = R_{\mathcal{G}}(\alpha)\, R_{\Lambda\infty}(\alpha)$ yield

$$Z = \hat{Z} + R_{\Lambda\infty}(\alpha)\, \hat{X}^{\dagger T}(\alpha)\, \hat{N}\hat{M}^{-1}\hat{X}^{\dagger}(\alpha)\, R_{\Lambda\infty}(\alpha). \tag{7.7}$$

In turn, by using (7.6) and (7.2) we have $Z = \hat{Z}$ and hence formula (7.7) becomes

$$R_{\Lambda\infty}(\alpha)\, \hat{X}^{\dagger T}(\alpha)\, \hat{N}\hat{M}^{-1}\hat{X}^{\dagger}(\alpha)\, R_{\Lambda\infty}(\alpha) = 0. \tag{7.8}$$

From Remark 3.7 it follows that $\operatorname{Im} \hat{X}^{\dagger}(\alpha)\, R_{\Lambda\infty}(\alpha) = \operatorname{Im} P_{\hat{\mathcal{S}}\infty}$, which means that we have $\hat{X}^{\dagger}(\alpha)\, R_{\Lambda\infty}(\alpha)\, K = P_{\hat{\mathcal{S}}\infty}$ for some invertible matrix K. By using (7.8) we then obtain that

$$P_{\hat{\mathcal{S}}\infty}\, \hat{N}\hat{M}^{-1} P_{\hat{\mathcal{S}}\infty} = K^T R_{\Lambda\infty}(\alpha)\, \hat{X}^{\dagger T}(\alpha)\, \hat{N}\hat{M}^{-1}\hat{X}^{\dagger}(\alpha)\, R_{\Lambda\infty}(\alpha)\, K \overset{(7.8)}{=} 0,$$

which is the last condition in (7.4). Thus, according to Proposition 3.14 the conjoined basis (X, U) is a principal solution of (H) at infinity with respect to $[\alpha, \infty)$. Finally, with the aid of Remark 7.6 the matrix $Q(t)$ is then a distinguished solution of ($\mathcal{R}$) at infinity with respect to $[\alpha, \infty)$. The proof is complete. $\square$

In the next three results we study the minimality of distinguished solutions of ($\mathcal{R}$). This minimality property needs to be understood in the following sense. For every symmetric solution $Q(t)$ of ($\mathcal{R}$) there exists a distinguished solution of ($\mathcal{R}$), which exists on the same interval and is at the same time smaller than $Q(t)$ on this interval (Theorems 7.16 and 7.18). On the other hand, any symmetric solution of (H), which is smaller than a distinguished solution of (H) on some interval, is a distinguished solution itself with respect to this interval (Theorem 7.20). However, in general there is no universal "smallest" distinguished solution of the Riccati equation ($\mathcal{R}$), see Remark 7.21. We also note that in the first result we consider the case when the solutions satisfy condition (1.4), while in the second and third result this assumption is removed.

Theorem 7.16. *Assume* (1.1). *Let $\mathcal{G}$ be a genus of conjoined bases of* (H) *with the matrix $R_{\mathcal{G}}(t)$ in* (3.11) *and let $Q(t)$ be a symmetric solution of the Riccati equation* ($\mathcal{R}$) *on $[\alpha,\infty)\subseteq[\alpha_\infty,\infty)$ such that inclusion* (1.4) *holds. Then there exists a distinguished solution $\hat{Q}(t)$ of* ($\mathcal{R}$) *at infinity with respect to $[\alpha,\infty)$ satisfying* (1.4) *such that $Q(t)\geq\hat{Q}(t)$ for every $t\in[\alpha,\infty)$.*

Proof. Let $Q(t)$ be as in the theorem and let (X,U) be its associated conjoined basis of (H) in Theorem 4.21 or Remark 4.23. Then (X,U) has constant kernel on $[\alpha,\infty)$ and belongs to the genus $\mathcal{G}$. Moreover, through (1.4) the matrix $Q(t)=R_{\mathcal{G}}(t)\,Q(t)\,R_{\mathcal{G}}(t)$ is the Riccati quotient in (2.7) corresponding to (X,U) on $[\alpha,\infty)$. Let T_α be the T-matrix in (2.13) associated with (X,U) on $[\alpha,\infty)$ and consider the solution $(\hat{X},\hat{U})$ of (H) in (3.4). From Theorem 3.5 and Remark 3.6 we know that $(\hat{X},\hat{U})$ is a principal solution of (H) at infinity belonging to the genus $\mathcal{G}$. Let $\hat{Q}(t)$ be its corresponding Riccati quotient in (2.7) on $[\alpha,\infty)$. According to Theorem 7.5 and Remark 7.6 the matrix $\hat{Q}(t)$ is a distinguished solution of ($\mathcal{R}$) at infinity with respect to $[\alpha,\infty)$ satisfying condition (1.4). Moreover, since the matrix T_α is nonnegative definite, by Remark 2.5, with the aid of Corollary 5.5(ii) with $\tilde{Q}:=\hat{Q}$, $N:=-T_\alpha$, and $M:=I$ we then have that $Q(\alpha)\geq\hat{Q}(\alpha)$. Finally, this inequality implies through Corollary 4.15 that $Q(t)\geq\hat{Q}(t)$ for all $t\in[\alpha,\infty)$, which completes the proof. □

Remark 7.17. (i) By applying (5.7) in Theorem 5.3 we obtain an exact relation between the Riccati quotients $Q(t)$ and $\hat{Q}(t)$ on $[\alpha,\infty)$. Namely, the formula

$$\hat{Q}(t)=Q(t)-X^{\dagger T}(t)\,T_\alpha\,[P-S_\alpha(t)\,T_\alpha]^\dagger\,X^\dagger(t) \tag{7.9}$$

holds for every $t\in[\alpha,\infty)$. In particular, for $t=\alpha$ the equality in (7.9) becomes

$$\hat{Q}(\alpha)=Q(\alpha)-X^{\dagger T}(\alpha)\,T_\alpha\,X^\dagger(\alpha). \tag{7.10}$$

(ii) According to Remark 7.14, the point $\alpha\in[\alpha_\infty,\infty)$ in Theorem 7.16 satisfies $\alpha>\hat{\alpha}_{\min}$. Moreover, from Theorems 4.3 and 7.16 it follows that the last inequality holds even when condition (1.4) regarding the matrix $Q(t)$ is dropped. Hence, we conclude that for any genus $\mathcal{G}$ the open interval $(\hat{\alpha}_{\min},\infty)$ is the maximal set such that there exists a symmetric solution of the Riccati equation ($\mathcal{R}$) on $(\hat{\alpha}_{\min},\infty)$.

Theorem 7.18. *Assume* (1.1). *Let $\mathcal{G}$ be a genus of conjoined bases of* (H) *with the orthogonal projector $R_{\mathcal{G}}(t)$ in* (3.11). *Let $Q(t)$ be a symmetric solution of the Riccati equation* ($\mathcal{R}$) *on $[\alpha,\infty)\subseteq[\alpha_\infty,\infty)$. Then there exists a distinguished solution $\hat{Q}(t)$ of* ($\mathcal{R}$) *with respect to $[\alpha,\infty)$ such that $Q(t)\geq\hat{Q}(t)$ holds for every $t\in[\alpha,\infty)$.*

Proof. We proceed similarly as in the proof of Theorem 7.16. Let (X,U) be a conjoined basis of (H) from the genus $\mathcal{G}$, which corresponds to $Q(t)$ on $[\alpha,\infty)$ through

Theorem 4.21. In particular, (X, U) has constant kernel on $[\alpha, \infty)$ and the symmetric matrix $Q_*(t) := R_{\mathcal{G}}(t)\, Q(t)\, R_{\mathcal{G}}(t)$ is the associated Riccati quotient in (2.7) for every $t \in [\alpha, \infty)$. Moreover, according to formula (7.10) in Remark 7.17 with $Q_* := Q$ the solution $\hat{Q}_*(t)$ of $(\mathcal{R})$ satisfying the condition

$$\hat{Q}_*(\alpha) = Q_*(\alpha) - X^{\dagger T}(\alpha)\, T_\alpha\, X^\dagger(\alpha) = R_{\mathcal{G}}(\alpha)\, Q(\alpha)\, R_{\mathcal{G}}(\alpha) - X^{\dagger T}(\alpha)\, T_\alpha\, X^\dagger(\alpha) \quad (7.11)$$

is a distinguished solution of $(\mathcal{R})$ at infinity with respect to $[\alpha, \infty)$. Here T_α is the T-matrix in (2.13) associated with (X, U). Furthermore, let D_α be the matrix in (4.19), which corresponds to $Q(t)$ through the F-matrix $F_\alpha(t)$ in (4.18) on $[\alpha, \infty)$, and consider the symmetric solution $\hat{Q}(t)$ of $(\mathcal{R})$ given by initial condition $\hat{Q}(\alpha) := Q(\alpha) - D_\alpha$. We will show that $\hat{Q}(t)$ is a distinguished solution of $(\mathcal{R})$ at infinity with respect to $[\alpha, \infty)$. Let $R_{\Lambda\infty}(t)$ be the orthogonal projector defined in (3.7). Similarly, as in the proof of Theorem 7.15 we will use the notation

$$\hat{Z} := R_{\Lambda\infty}(\alpha)\, \hat{Q}(\alpha)\, R_{\Lambda\infty}(\alpha), \quad \hat{Z}_* := R_{\Lambda\infty}(\alpha)\, \hat{Q}_*(\alpha)\, R_{\Lambda\infty}(\alpha). \quad (7.12)$$

With the aid of (3.11) and Remark 5.2(ii) together with the symmetry of the matrices $R_{\Lambda\infty}(t)$, $R_{\mathcal{G}}(t)$, and D_α we have the identities $R_{\Lambda\infty}(\alpha)\, R_{\mathcal{G}}(\alpha) = R_{\Lambda\infty}(\alpha) = R_{\mathcal{G}}(\alpha)\, R_{\Lambda\infty}(\alpha)$ and $R_{\Lambda\infty}(\alpha)\, D_\alpha = D_\alpha = D_\alpha\, R_{\Lambda\infty}(\alpha)$. By combining these properties with (7.11) and the second equality in (5.6) we obtain that

$$\begin{aligned}
\hat{Z} &\overset{(7.12)}{=} R_{\Lambda\infty}(\alpha)\, \hat{Q}(\alpha)\, R_{\Lambda\infty}(\alpha) = R_{\Lambda\infty}(\alpha)\, [\, Q(\alpha) - D_\alpha]\, R_{\Lambda\infty}(\alpha) \\
&= R_{\Lambda\infty}(\alpha)\, R_{\mathcal{G}}(\alpha)\, Q(\alpha)\, R_{\mathcal{G}}(\alpha)\, R_{\Lambda\infty}(\alpha) - D_\alpha \\
&\overset{(7.11)}{=} R_{\Lambda\infty}(\alpha)\, [\, \hat{Q}_*(\alpha) + X^{\dagger T}(\alpha)\, T_\alpha\, X^\dagger(\alpha)\,]\, R_{\Lambda\infty}(\alpha) - D_\alpha \\
&\overset{(7.12)}{=} \hat{Z}_* + R_{\Lambda\infty}(\alpha)\, X^{\dagger T}(\alpha)\, T_\alpha\, X^\dagger(\alpha)\, R_{\Lambda\infty}(\alpha) - D_\alpha \overset{(5.6)}{=} \hat{Z}_*.
\end{aligned} \quad (7.13)$$

Finally, since the point $\alpha > \hat{\alpha}_{\min}$ by Remark 7.17(ii), from (7.12), the equation in (7.13), and Theorem 7.15 it follows immediately that the solution $\hat{Q}(t)$ is a distinguished solution of $(\mathcal{R})$ at infinity with respect to $[\alpha, \infty)$. In particular, the matrix $\hat{Q}(t)$ solves equation $(\mathcal{R})$ on the whole interval $[\alpha, \infty)$. And since $Q(\alpha) - \hat{Q}(\alpha) = D_\alpha \geq 0$, we conclude by Corollary 4.14 that the inequality $Q(t) \geq \hat{Q}(t)$ holds for every $t \in [\alpha, \infty)$, which completes the proof. ☐

Remark 7.19. We note that the converse to Theorem 7.18 also holds. More precisely, if $\hat{Q}(t)$ is a distinguished solution of $(\mathcal{R})$ at infinity with respect to the interval $[\alpha, \infty)$, then every symmetric solution $Q(t)$ of $(\mathcal{R})$, which satisfies the condition $Q(\alpha) \geq \hat{Q}(\alpha)$, exists on the whole interval $[\alpha, \infty)$ and the inequality $Q(t) \geq \hat{Q}(t)$ holds on $[\alpha, \infty)$. This observation is a direct application of Corollary 4.15 with the choice $Q := \hat{Q}$ and $\tilde{Q} := Q$.

Theorem 7.20. *Assume* (1.1) *and let* $\mathcal{G}$ *be a genus of conjoined bases of* (H) *with the matrix* $R_{\mathcal{G}}(t)$ *in* (3.11). *Let* $\tilde{Q}(t)$ *be a distinguished solution of the Riccati equation* $(\mathcal{R})$ *with respect to the interval* $[\alpha, \infty) \subseteq [\alpha_\infty, \infty)$. *Moreover, let* $Q(t)$ *be a symmetric solution of* $(\mathcal{R})$ *on* $[\alpha, \infty)$ *satisfying the initial condition* $\tilde{Q}(\alpha) \geq Q(\alpha)$. *Then* $Q(t)$ *is a distinguished solution of* $(\mathcal{R})$ *at infinity with respect to* $[\alpha, \infty)$ *and the inequality* $\tilde{Q}(t) \geq Q(t)$ *holds for all* $t \in [\alpha, \infty)$.

Proof. Let $\tilde{Q}(t)$ and $Q(t)$ be as in the theorem. By using Corollary 4.14 we obtain the inequality $\tilde{Q}(t) \geq Q(t)$ on $[\alpha, \infty)$. On the other hand, according to Theorem 7.18 there exists a distinguished solution $\hat{Q}(t)$ of $(\mathcal{R})$ at infinity such that $Q(t) \geq \hat{Q}(t)$

for every $t \in [\alpha, \infty)$. Hence, for $t = \alpha$ we have the relations $\tilde{Q}(\alpha) \geq Q(\alpha) \geq \hat{Q}(\alpha)$. Consequently, by multiplying the last inequalities by the matrix $R_{\Lambda\infty}(\alpha)$ defined in (3.7) from the both sides we obtain that

$$R_{\Lambda\infty}(\alpha)\,\tilde{Q}(\alpha)\,R_{\Lambda\infty}(\alpha) \geq R_{\Lambda\infty}(\alpha)\,Q(\alpha)\,R_{\Lambda\infty}(\alpha) \geq R_{\Lambda\infty}(\alpha)\,\hat{Q}(\alpha)\,R_{\Lambda\infty}(\alpha). \tag{7.14}$$

But Theorem 7.15 and the fact that both the solutions $\tilde{Q}(t)$ and $\hat{Q}(t)$ are distinguished with respect to $[\alpha, \infty)$ yield $R_{\Lambda\infty}(\alpha)\,\tilde{Q}(\alpha)\,R_{\Lambda\infty}(\alpha) = R_{\Lambda\infty}(\alpha)\,\hat{Q}(\alpha)\,R_{\Lambda\infty}(\alpha)$. Therefore, the inequalities in (7.14) are implemented as the equalities. In turn, applying Theorem 7.15 once more then implies that $Q(t)$ is a distinguished solution of ($\mathcal{R}$) at infinity with respect to $[\alpha, \infty)$ as well. The proof is complete. $\square$

Remark 7.21. Given a genus $\mathcal{G}$ of conjoined bases of (H) with the matrix $R_{\mathcal{G}}(t)$ defined in (3.11), let $\hat{Q}(t)$ be a distinguished solution of ($\mathcal{R}$) at infinity with respect to the interval $[\alpha, \infty) \subseteq [\alpha_\infty, \infty)$. Then there exist distinguished solutions $\hat{Q}_*(t)$ and $\hat{Q}_{**}(t)$ of ($\mathcal{R}$) satisfying

$$\hat{Q}_*(t) \leq \hat{Q}(t) \leq \hat{Q}_{**}(t), \quad t \in [\alpha, \infty). \tag{7.15}$$

The solutions $\hat{Q}_*(t)$ and $\hat{Q}_{**}(t)$ are given, for example, by the initial conditions

$$\hat{Q}_*(\alpha) = \hat{Q}(\alpha) - I + R_{\Lambda\infty}(\alpha) \quad \text{and} \quad \hat{Q}_{**}(\alpha) = \hat{Q}(\alpha) + I - R_{\Lambda\infty}(\alpha), \tag{7.16}$$

where $R_{\Lambda\infty}(t)$ is the orthogonal projector defined in (3.7). Indeed, by using (7.16) and the basic properties of orthogonal projectors from Section 2 and by utilizing the notation in (7.6) we get

$$R_{\Lambda\infty}(\alpha)\,\hat{Q}_*(\alpha)\,R_{\Lambda\infty}(\alpha) \overset{(7.16)}{=} R_{\Lambda\infty}(\alpha)\,[\,\hat{Q}(\alpha) - I + R_{\Lambda\infty}(\alpha)\,]\,R_{\Lambda\infty}(\alpha) \overset{(7.6)}{=} \hat{Z},$$
$$R_{\Lambda\infty}(\alpha)\,\hat{Q}_{**}(\alpha)\,R_{\Lambda\infty}(\alpha) \overset{(7.16)}{=} R_{\Lambda\infty}(\alpha)\,[\,\hat{Q}(\alpha) + I - R_{\Lambda\infty}(\alpha)\,]\,R_{\Lambda\infty}(\alpha) \overset{(7.6)}{=} \hat{Z}.$$

From Theorem 7.13 we know that $\alpha > \hat{\alpha}_{\min}$. Hence, by applying Theorem 7.15 we obtain immediately that $\hat{Q}_*(t)$ and $\hat{Q}_{**}(t)$ are distinguished solutions of ($\mathcal{R}$) at infinity with respect to $[\alpha, \infty)$. In addition, since $\hat{Q}(\alpha) - \hat{Q}_*(\alpha) = I - R_{\Lambda\infty}(\alpha) = \hat{Q}_{**}(\alpha) - \hat{Q}(\alpha)$ by (7.16) and $I - R_{\Lambda\infty}(\alpha) \geq 0$, we have the inequalities $\hat{Q}_*(\alpha) \leq \hat{Q}(\alpha) \leq \hat{Q}_{**}(\alpha)$. In turn, according to Corollary 4.14 the inequalities in (7.15) hold. This observation shows that for the case of a general (not necessarily controllable) system (H) the partially ordered set of all distinguished solutions of ($\mathcal{R}$) has neither a minimal element nor a maximal element.

The considerations in Theorems 7.15 and 7.16 show that for the minimal genus $\mathcal{G}_{\min}$, i.e., for $R_{\mathcal{G}}(t) = R_{\Lambda\infty}(t)$, there exists a uniquely determined distinguished solution of the Riccati equation ($\mathcal{R}$) with

$$\left.\begin{aligned} \mathcal{A}(t) &:= A(t)R_{\Lambda\infty}(t) - A^T(t)\,[I - R_{\Lambda\infty}(t)], \\ \mathcal{B}(t) &:= B(t), \quad \mathcal{C}(t) := R_{\Lambda\infty}(t)\,C(t)\,R_{\Lambda\infty}(t), \end{aligned}\right\} \tag{7.17}$$

which represents the smallest element in the set of all symmetric solutions $Q(t)$ of equation ($\mathcal{R}$) satisfying (1.4).

Definition 7.22. Let $\mathcal{G}_{\min}$ be the minimal genus of conjoined bases of system (H) with the minimal orthogonal projector $R_{\Lambda\infty}(t)$ in (3.7). A symmetric solution $\hat{Q}(t)$ of the Riccati equation ($\mathcal{R}$) with the coefficients in (7.17) is said to be a *minimal distinguished solution at infinity* if the matrix $\hat{Q}(t)$ is defined on some interval

$[\alpha, \infty) \subseteq [\alpha_\infty, \infty)$ such that

$$\operatorname{Im} \hat{Q}(t) \subseteq \operatorname{Im} R_{\Lambda\infty}(t), \quad t \in [\alpha, \infty), \tag{7.18}$$

and its corresponding matrix $\hat{F}_\alpha(t)$ in (4.18) satisfies $\hat{F}_\alpha^\dagger(t) \to 0$ as $t \to \infty$.

The following result shows the existence and uniqueness of the minimal distinguished solution of the Riccati equation ($\mathcal{R}$) for the minimal genus $\mathcal{G}_{\min}$, as well as its minimality property.

Theorem 7.23. *Assume* (1.1)*. Then system* (H) *is nonoscillatory if and only if there exists a minimal distinguished solution $\hat{Q}(t)$ of the Riccati equation* ($\mathcal{R}$) *with the coefficients in* (7.17)*. In this case, the minimal distinguished solution $\hat{Q}(t)$ is determined uniquely and any symmetric solution $Q(t)$ of* ($\mathcal{R}$) *on $[\alpha, \infty) \subseteq [\alpha_\infty, \infty)$ with* (7.18) *satisfies $Q(t) \geq \hat{Q}(t)$ on $[\alpha, \infty)$.*

Proof. The first part of the theorem coincides with the statement of Remark 7.10 for the genus $\mathcal{G} = \mathcal{G}_{\min}$ and for the corresponding orthogonal projector $R_\mathcal{G}(t)$ in (3.11) equal to the minimal orthogonal projector $R_{\Lambda\infty}(t)$ defined in (3.7). The uniqueness of the distinguished solution $\hat{Q}(t)$ follows from Theorem 7.15 with $\mathcal{G} := \mathcal{G}_{\min}$. More precisely, let $\hat{Q}(t)$ and $\hat{Q}_*(t)$ be two distinguished solutions of equation ($\mathcal{R}$) for the minimal genus $\mathcal{G}_{\min}$, which satisfy condition (1.4) (with $R_\mathcal{G}(t) = R_{\Lambda\infty}(t)$) on $[\alpha, \infty)$ for some $\alpha \geq \alpha_\infty$. From Theorem 7.13 it follows that both matrices $\hat{Q}(t)$ and $\hat{Q}_*(t)$ solve ($\mathcal{R}$) on the maximal open interval $(\hat{\alpha}_{\min}, \infty)$ and hence, the point $\alpha \in (\hat{\alpha}_{\min}, \infty)$. According to Corollary 4.5 we then obtain the inclusions $\operatorname{Im} \hat{Q}(t) \subseteq \operatorname{Im} R_{\Lambda\infty}(t)$ and $\operatorname{Im} \hat{Q}_*(t) \subseteq \operatorname{Im} R_{\Lambda\infty}(t)$ for every $t \in (\hat{\alpha}_{\min}, \infty)$. Consequently, combining these facts with the symmetry of $\hat{Q}(t)$ and $\hat{Q}_*(t)$ yields

$$R_{\Lambda\infty}(t)\, \hat{Q}(t)\, R_{\Lambda\infty}(t) = \hat{Q}(t), \quad R_{\Lambda\infty}(t)\, \hat{Q}_*(t)\, R_{\Lambda\infty}(t) = \hat{Q}_*(t) \tag{7.19}$$

for all $t \in (\hat{\alpha}_{\min}, \infty)$. Finally, by using formula (7.2) in Theorem 7.15 with $\mathcal{G} := \mathcal{G}_{\min}$ and $Q := \hat{Q}_*$ together with (7.19) we obtain on $(\hat{\alpha}_{\min}, \infty)$ that

$$\hat{Q}(t) \overset{(7.19)}{=} R_{\Lambda\infty}(t)\, \hat{Q}(t)\, R_{\Lambda\infty}(t) \overset{(7.2)}{=} R_{\Lambda\infty}(t)\, \hat{Q}_*(t)\, R_{\Lambda\infty}(t) \overset{(7.19)}{=} \hat{Q}_*(t).$$

Thus, the distinguished solutions $\hat{Q}(t)$ and $\hat{Q}_*(t)$ coincide. Finally, the minimality property of the minimal distinguished solution of ($\mathcal{R}$) at infinity follows from Theorem 7.16. Namely, by using the latter reference for any symmetric solution $Q(t)$ of ($\mathcal{R}$) on $[\alpha, \infty) \subseteq [\alpha_\infty, \infty)$ with (7.18) there exists a distinguished solution $\hat{Q}(t)$ of ($\mathcal{R}$) at infinity with respect to $[\alpha, \infty)$ satisfying (7.18) such that $Q(t) \geq \hat{Q}(t)$ for every $t \in [\alpha, \infty)$. In fact, the matrix $\hat{Q}(t)$ is the minimal distinguished solution of ($\mathcal{R}$) at infinity, by Definition 7.22. The proof is complete. $\square$

Remark 7.24. The minimal distinguished solution of ($\mathcal{R}$) at infinity in Theorem 7.23 will be denoted by $\hat{Q}_{\min}$. The minimal distinguished solution $\hat{Q}_{\min}$ plays for the theory of the Riccati differential equations ($\mathcal{R}$) or (R) a similar role as the minimal principal solution $(\hat{X}_{\min}, \hat{U}_{\min})$ of system (H) at infinity for the theory of principal solutions at infinity.

Remark 7.25. When system (H) is completely controllable, the main results of this section give the classical statements about the distinguished solutions at infinity of the Riccati equation (R). More precisely, the following holds.

- The results in Corollary 7.7 and Theorem 7.15 yield the correspondence between the unique principal solution of (H) at infinity and the unique distinguished solution of (R) at infinity, see [7, pg. 53] or [23, pp. 45–46].
- The result in Theorem 7.8 provides a characterization of the nonoscillation of system (H) in terms of the existence of the unique distinguished solution of (R) at infinity, see the necessary condition in [22, Theorem VII.3.3]. Note that the nonoscillation of (H) is defined in [22, Section VII.3] in terms of disconjugacy of (H), i.e., in terms of the nonexistence of mutually conjugate points, which is a stronger concept than the nonoscillation of (H) as we define in Section 2. We note also that the sufficiency part of Theorem 7.8 is new also in the completely controllable case.
- The results in Theorems 7.18 and 7.20 yield the minimality property of the unique distinguished solution of (R) at infinity, see [7, Theorem 8, pg. 54] or [23, Theorem IV.4.2].

Indeed, in this case $d_\infty = 0$ and there is only one minimal/maximal genus of conjoined bases of (H). This implies that $\alpha_\infty = a$ and the orthogonal projector $R_{\Lambda\infty}(t)$ in (3.7) satisfies $R_{\Lambda\infty}(t) \equiv I$ on $[a, \infty)$. Therefore, the unique Riccati equation ($\mathcal{R}$) associated with the minimal/maximal genus coincides with the classical Riccati equation (R). Moreover, under the Legendre condition (1.1) the nonoscillation of system (H) is then equivalent with the existence of a unique (minimal) distinguished solution $\hat{Q}$ of (R) at infinity. In addition, the matrix $\hat{Q}$ constitutes the smallest symmetric solution of the Riccati equation (R), that is, every symmetric solution Q of (R) on $[\alpha, \infty) \subseteq [a, \infty)$ satisfies inequality (1.5).

Remark 7.26. We note that the results commented on in Remark 7.25 hold under a weaker assumption than the complete controllability of (H). More precisely, under (1.1) and the nonoscillation of (H) the existence of a unique distinguished solution at infinity of a (unique) Riccati equation ($\mathcal{R}$) is equivalent with the fact that the maximal order of abnormality $d_\infty = 0$.

8. **Examples.**

In this section we provide several examples which illustrate the presented theory of Riccati equations for abnormal system (H).

Example 8.1. In the first example we explore a controllable linear Hamiltonian system. For $n = 1$, $a = 0$ we consider system (H) with $A(t) = 0$, $B(t) = 1 + t^2$, and $C(t) = -2/(1 + t^2)^2$. This system comes from the second order Sturm–Liouville equation $[y'/(1 + t^2)]' + 2y/(1 + t^2)^2 = 0$. The matrix $B(t) > 0$ on $[0, \infty)$, which implies that system (H) is completely controllable on $[0, \infty)$ with $d[0, \infty) = d_\infty = 0$ and $\alpha_\infty = 0$, by (2.19). Thus, there exists only one (minimal/maximal) genus $\mathcal{G}$ of conjoined bases with the corresponding orthogonal projector $R_{\mathcal{G}}(t) \equiv 1$ on $[0, \infty)$ and consequently, the unique Riccati equation

$$Q' + (1 + t^2)\, Q^2 + 2/(1 + t^2)^2 = 0, \quad t \in [0, \infty). \tag{8.1}$$

In [30, Example 7.1] we showed that system (H) is nonoscillatory and that the principal solutions at infinity are nonzero multiples of

$$\big(\hat{X}(t), \hat{U}(t)\big) = \big(t,\, 1/(1 + t^2)\big),$$

with $\hat{\alpha}_{\min} = 0$, by (3.1). Therefore, by Theorem 7.5 and Remark 7.25 the unique (minimal) distinguished solution $\hat{Q}$ of (8.1) at infinity satisfies

$$\hat{Q}(t) = 1/[t(1 + t^2)], \quad t \in (0, \infty).$$

Moreover, by using Proposition 4.12 with $Q := \hat{Q}$ the general solution $Q(\cdot, \alpha, p)$ of the Riccati equation (8.1) defined on an interval $[\alpha, \infty) \subseteq (0, \infty)$ has the form

$$Q(t, \alpha, p) = \hat{Q}(t) + p/[t^2 + pt(t - \alpha)(t + 1/\alpha)], \quad t \in [\alpha, \infty), \tag{8.2}$$

with $p \in [0, \infty) \cup \{\infty\}$, where

$$Q(t, \alpha, \infty) := \lim_{p \to \infty} Q(t, \alpha, p) \overset{(8.2)}{=} \hat{Q}(t) + 1/[t(t - \alpha)(t + 1/\alpha)], \quad t \in [\alpha, \infty). \tag{8.3}$$

From formulas (8.2)–(8.3) it then follows that for any point $\alpha > 0$ and parameter $p \in [0, \infty) \cup \{\infty\}$ we have the inequality $Q(t, \alpha, p) \geq \hat{Q}(t)$ on $[\alpha, \infty)$, as we claim in Theorem 7.18 or Remark 7.25.

Example 8.2. In this example we consider the so-called zero system (H) with $n \times n$ coefficient matrices $A(t) = B(t) = C(t) \equiv 0$ on $[a, \infty)$. This system is nonoscillatory and extremely abnormal, that is, $d[a, \infty) = d_\infty = n$ and hence, $\alpha_\infty = a$ and $R_{\Lambda\infty}(t) \equiv 0$ on $[a, \infty)$. In [29, Example 8.2] and [31, Example 5.7] we showed that every conjoined basis of (H) is a constant principal solution at infinity with respect to $[a, \infty)$ and that the set of all genera of conjoined bases of (H) is isomorphic to the complete lattice of all subspaces in $\mathbb{R}^n$. Namely, for every constant orthogonal projector $R \in \mathbb{R}^{n \times n}$ there exists a unique genus $\mathcal{G}$ of conjoined bases of (H) such that $R_{\mathcal{G}}(t) \equiv R$ on $[a, \infty)$. The associated Riccati equation ($\mathcal{R}$) reduces to $Q' = 0$. In this case, every symmetric solution of ($\mathcal{R}$) is a constant distinguished solution at infinity with respect to the interval $[a, \infty)$, so that $\hat{\alpha}_{\min} = a$. In particular, for any constant symmetric matrix $M \in \mathbb{R}^{n \times n}$ the pair $(\hat{X}, \hat{U}) := (R, MR + I - R)$ constitutes a constant principal solution of (H) at infinity belonging to $\mathcal{G}$, which corresponds to the distinguished solution $\hat{Q}(t) \equiv M$ of ($\mathcal{R}$) on $[a, \infty)$ via Theorems 4.21 and 7.4. Moreover, the minimal distinguished solution at infinity satisfies $\hat{Q}_{\min}(t) \equiv 0$ on $[a, \infty)$.

In the previous two examples we studied the situation when system (H) possessed only one Riccati equation ($\mathcal{R}$). However, this will be not the case of the system presented in the last example.

Example 8.3. For $n = 3$ and $a = 0$ we consider system (H) with the coefficients $A(t) = \operatorname{diag}\{0, 0, 1\}$, $B(t) = \operatorname{diag}\{1 + t^2, 0, 0\}$, and $C(t) = \operatorname{diag}\{-2/(1 + t^2)^2, 0, 0\}$ on $[0, \infty)$. In this case we have $d_\infty = 2$, $\alpha_\infty = 0$, and $R_{\Lambda\infty}(t) \equiv \operatorname{diag}\{1, 0, 0\}$ on $[0, \infty)$. Moreover, in [31, Example 5.8] we examined the set of all genera $\mathcal{G}$ of conjoined bases of (H) and found a principal solution at infinity in each genus $\mathcal{G}$. We will continue in this study by illustrating the concept of distinguished solutions at infinity of the associated Riccati equations ($\mathcal{R}$). For the minimal genus $\mathcal{G} = \mathcal{G}_{\min}$ represented by the orthogonal projector $R_{\mathcal{G}}(t) = R_{\Lambda\infty}(t)$ on $[0, \infty)$ we have the Riccati equation ($\mathcal{R}$) with the coefficients in (7.17), i.e.,

$$\mathcal{A}_{\min}(t) = -A(t), \quad \mathcal{B}_{\min}(t) = B(t), \quad \mathcal{C}_{\min}(t) = C(t), \quad t \in [0, \infty).$$

This Riccati equation possesses on $(0, \infty)$ the minimal distinguished solution

$$\hat{Q}_{\min}(t) = \operatorname{diag}\{1/[t(1 + t^2)], 0, 0\} \tag{8.4}$$

constructed from the minimal principal solution

$$\big(\hat{X}_{\min}(t), \hat{U}_{\min}(t)\big) = \big(\operatorname{diag}\{t, 0, 0\}, \operatorname{diag}\{1/(1 + t^2), 1, \mathrm{e}^{-t}\}\big)$$

of (H) at infinity via Theorem 4.18, as well as the distinguished solution

$$\hat{Q}_0(t) = \operatorname{diag}\{1/[t(1 + t^2)], 1, -\mathrm{e}^{2t}\}, \tag{8.5}$$

which does not satisfy condition (7.18). In particular, the distinguished solutions in (8.4) and (8.5) are mutually incomparable on the interval $(0,\infty)$. Similarly, with the maximal genus $\mathcal{G}=\mathcal{G}_{\max}$ represented by the orthogonal projector $R_{\mathcal{G}}(t)\equiv I$ on $[0,\infty)$ there is associated the Riccati equation (R) with the pair of incomparable distinguished solutions at infinity

$$\hat{Q}(t)=\operatorname{diag}\{1/[t(1+t^2)],-1,\mathrm{e}^{-2t}\},\quad \hat{Q}_*(t)=\operatorname{diag}\{1/[t(1+t^2)],1,-\mathrm{e}^{-2t}\} \tag{8.6}$$

for $t\in(0,\infty)$. We note that $\hat{Q}(t)$ and $\hat{Q}_*(t)$ in (8.6) are both the Riccati quotients in (2.7), which correspond to the maximal principal solutions

$$\big(\hat{X}(t),\hat{U}(t)\big)=\big(\operatorname{diag}\{t,1,\mathrm{e}^{t}\},\,\operatorname{diag}\{1/(1+t^2),-1,\mathrm{e}^{-t}\}\big),$$
$$\big(\hat{X}_*(t),\hat{U}_*(t)\big)=\big(\operatorname{diag}\{t,1,\mathrm{e}^{t}\},\,\operatorname{diag}\{1/(1+t^2),1,-\mathrm{e}^{-t}\}\big)$$

of (H) at infinity, respectively. In the remaining part of this example we analyze for three different genera with rank equal to $r=2$ the corresponding Riccati equations $(\mathcal{R})$ and their distinguished solutions. More precisely, according to [31, Example 5.8] we consider the genera $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$ given by

$$R_{\mathcal{G}_1}(t)\equiv\operatorname{diag}\{1,0,1\},\quad R_{\mathcal{G}_2}(t)\equiv\operatorname{diag}\{1,1,0\},$$
$$R_{\mathcal{G}_3}(t)=\frac{1}{\mathrm{e}^{2t}+1}\begin{pmatrix}\mathrm{e}^{2t}+1&0&0\\0&1&\mathrm{e}^{t}\\0&\mathrm{e}^{t}&\mathrm{e}^{2t}\end{pmatrix}$$

on $[0,\infty)$. With the genus $\mathcal{G}_1$ we associate the Riccati equation $(\mathcal{R})$ with

$$\mathcal{A}_1(t)=A(t),\quad \mathcal{B}_1(t)=B(t),\quad \mathcal{C}_1(t)=C(t),\quad t\in[0,\infty),$$

possessing the pair of incomparable distinguished solutions at infinity

$$\hat{Q}_1(t)=\operatorname{diag}\{1/[t(1+t^2)],0,-\mathrm{e}^{-2t}\},\quad \hat{Q}_{1*}(t)=\operatorname{diag}\{1/[t(1+t^2)],-1,0\}$$

for $(0,\infty)$. The matrix $\hat{Q}_1(t)$ is the Riccati quotient in (2.7), which corresponds to the principal solution

$$\big(\hat{X}_1(t),\hat{U}_1(t)\big)=\big(\operatorname{diag}\{t,0,\mathrm{e}^{t}\},\,\operatorname{diag}\{1/(1+t^2),1,-\mathrm{e}^{-t}\}\big)$$

of (H) at infinity belonging to $\mathcal{G}_1$, while the distinguished solution $\hat{Q}_{1*}(t)$ does not satisfy (1.4). Similarly, for the genus $\mathcal{G}_2$ we have the Riccati equation $(\mathcal{R})$ with

$$\mathcal{A}_2(t)=-A(t),\quad \mathcal{B}_2(t)=B(t),\quad \mathcal{C}_2(t)=C(t),\quad t\in[0,\infty),$$

which has the pair of incomparable distinguished solutions at infinity

$$\hat{Q}_2(t)=\operatorname{diag}\{1/[t(1+t^2)],1,0\},\quad \hat{Q}_{2*}(t)=\operatorname{diag}\{1/[t(1+t^2)],0,\mathrm{e}^{2t}\}$$

for $(0,\infty)$. The matrix $\hat{Q}_2(t)$ is the Riccati quotient in (2.7), which corresponds to the principal solution

$$\big(\hat{X}_2(t),\hat{U}_2(t)\big)=\big(\operatorname{diag}\{t,1,0\},\,\operatorname{diag}\{1/(1+t^2),1,\mathrm{e}^{-t}\}\big)$$

of (H) at infinity from the genus $\mathcal{G}_2$ and the distinguished solution $\hat{Q}_{2*}(t)$ does not satisfy (1.4). Finally, for the genus $\mathcal{G}_3$ we obtain the Riccati equation $(\mathcal{R})$ with

$$\mathcal{A}_3(t)=\frac{1}{\mathrm{e}^{2t}+1}\begin{pmatrix}0&0&0\\0&0&0\\0&2\mathrm{e}^{t}&\mathrm{e}^{2t}-1\end{pmatrix},\quad \mathcal{B}_3(t)=B(t),\quad \mathcal{C}_3(t)=C(t),\quad t\in[0,\infty),$$

having on $(0, \infty)$ the pair of incomparable distinguished solutions at infinity

$$\hat{Q}_3(t) = \frac{1}{(\mathrm{e}^{2t}+1)^2} \begin{pmatrix} (\mathrm{e}^{2t}+1)^2/[t(1+t^2)] & 0 & 0 \\ 0 & 1 & \mathrm{e}^t \\ 0 & \mathrm{e}^t & \mathrm{e}^{2t} \end{pmatrix},$$

$$\hat{Q}_{3*}(t) = \frac{1}{(\mathrm{e}^{2t}+1)^2} \begin{pmatrix} (\mathrm{e}^{2t}+1)^2/[t(1+t^2)] & 0 & 0 \\ 0 & 2\mathrm{e}^{2t}+3 & \mathrm{e}^{3t}+2\mathrm{e}^t \\ 0 & \mathrm{e}^{3t}+2\mathrm{e}^t & \mathrm{e}^{2t} \end{pmatrix}.$$

In particular, the matrix $\hat{Q}_3(t)$ constitutes the Riccati quotient in (2.7) associated with the principal solution at infinity

$$\big(\hat{X}_3(t), \hat{U}_3(t)\big) = \left(\begin{pmatrix} t & 0 & 0 \\ 0 & 1 & -1 \\ 0 & \mathrm{e}^t & -\mathrm{e}^t \end{pmatrix}, \begin{pmatrix} 1/(1+t^2) & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 2\mathrm{e}^{-t} & -2\mathrm{e}^{-t} \end{pmatrix} \right)$$

and the distinguished solution $\hat{Q}_{3*}(t)$ does not satisfy condition (1.4).

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APPENDIX C

Paper by Šepitka & Šimon Hilscher (JDE 2017)

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Comparative index and Sturmian theory for linear Hamiltonian systems ☆

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Abstract

The comparative index was introduced by J. Elyseeva (2007) as an efficient tool in matrix analysis, which has fundamental applications in the discrete oscillation theory. In this paper we implement the comparative index into the theory of continuous time linear Hamiltonian systems, study its properties, and apply it to obtain new Sturmian separation theorems as well as new and optimal estimates for left and right proper focal points of conjoined bases of these systems on bounded intervals. We derive our results for general possibly abnormal (or uncontrollable) linear Hamiltonian systems. The results turn out to be new even in the case of completely controllable systems. We also provide several examples, which illustrate our new theory.

MSC: 34C10

Keywords: Linear Hamiltonian system; Sturmian separation theorem; Proper focal point; Comparative index; Conjoined basis; Controllability

1. Introduction

In this paper we study oscillation properties of solutions of the linear Hamiltonian system

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$$x' = A(t)\,x + B(t)\,u, \quad u' = C(t)\,x - A^T(t)\,u, \qquad t \in [a,b], \tag{H}$$

where $A, B, C : [a,b] \to \mathbb{R}^{n\times n}$ are given piecewise continuous matrix-valued functions on the interval $[a,b]$ such that $B(t)$ and $C(t)$ are symmetric and the Legendre condition holds, i.e.,

$$B(t) \geq 0 \quad \text{for all } t \in [a,b]. \tag{1.1}$$

Here $n \in \mathbb{N}$ is a given dimension and $a, b \in \mathbb{R}$, $a < b$, are fixed numbers. The main results of this paper are concerned with the Sturmian type separation theorems about the number of focal points of conjoined bases of (H) in the given interval. We present a novel approach to this problem, which is based on the so-called comparative index of two conjoined bases of (H), see (2.3) in Section 2 below.

System (H) is traditionally studied under the complete controllability assumption. This means that the only solution (x,u) of (H) with $x(t) \equiv 0$ on a subinterval of $[a,b]$ with positive length is the trivial solution $(x,u) \equiv 0$ on $[a,b]$, see e.g. [5,17,22,26,27]. In this case $t_0 \in [a,b]$ is a *focal point* of a conjoined basis (X,U) of (H) if $X(t_0)$ is singular, and then

$$m(t_0) := \operatorname{def} X(t_0) = \dim \operatorname{Ker} X(t_0)$$

is its multiplicity. We refer to Section 3 for the definition of a conjoined basis. Every conjoined basis of (H) then has finitely many focal points in $[a,b]$, and the numbers of focal points in $(a,b]$ or in $[a,b)$ of any two conjoined bases of (H) differ by at most n, see [22, Theorem 4.1.3, p. 126] and [26, Corollary 1, p. 366]. In addition, for one conjoined basis (X,U) of (H) the difference between the numbers of its focal points in $(a,b]$ and in $[a,b)$ equals the value

$$\operatorname{def} X(b) - \operatorname{def} X(a) = \operatorname{rank} X(a) - \operatorname{rank} X(b). \tag{1.2}$$

When the controllability assumption is absent, Kratz showed in [23, Theorem 3] the following crucial result.

Proposition 1.1. *Assume that (1.1) holds. Then for any conjoined basis (X,U) of (H) the kernel of $X(t)$ is piecewise constant on $[a,b]$, i.e., there is a partition $a = t_0 < t_1 < \cdots < t_m = b$ such that* $\operatorname{Ker} X(t)$ *is constant on the open interval (t_j, t_{j+1}) for all $j \in \{0,1,\dots,m-1\}$ and*

$$\operatorname{Ker} X(t_j^-) \subseteq \operatorname{Ker} X(t_j), \qquad j \in \{1,2,\dots,m\}, \tag{1.3}$$

$$\operatorname{Ker} X(t_j^+) \subseteq \operatorname{Ker} X(t_j), \qquad j \in \{0,1,\dots,m-1\}. \tag{1.4}$$

The quantity $\operatorname{Ker} X(t_j^\pm)$ denotes the limit of the constant set $\operatorname{Ker} X(t)$ as $t \to t_j^\pm$. The inclusions in (1.3) and (1.4) follow from the continuity of $X(t)$ on $[a,b]$. In the subsequent work [36], Wahrheit defined the point $t_0 \in (a,b]$ to be a *left proper focal point* of (X,U) if $\operatorname{Ker} X(t_0^-) \subsetneqq \operatorname{Ker} X(t_0)$, with the multiplicity

$$m_L(t_0) := \operatorname{def} X(t_0) - \operatorname{def} X(t_0^-) = \operatorname{rank} X(t_0^-) - \operatorname{rank} X(t_0). \tag{1.5}$$

In a similar way we define $t_0 \in [a,b)$ to be a right proper focal point of (X,U) by the condition $\operatorname{Ker} X(t_0^+) \subsetneqq \operatorname{Ker} X(t_0)$, with the multiplicity

$$m_R(t_0) := \operatorname{def} X(t_0) - \operatorname{def} X(t_0^+) = \operatorname{rank} X(t_0^+) - \operatorname{rank} X(t_0). \tag{1.6}$$

The notations def $X(t_0^\pm)$ and rank $X(t_0^\pm)$ represent the one-sided limits at t_0 of the piecewise constant quantities def $X(t)$ and rank $X(t)$.

The above mentioned Sturmian separation theorem by Reid in [26, Corollary 1, p. 366] was generalized to possibly abnormal (or uncontrollable) system (H) in [24, Corollary 4.8] and [34, Theorems 1.4 and 1.5] by Kratz and the second author. We review these statements below for our future reference in this paper. Recall that a principal solution $(\hat{X}_s, \hat{U}_s)$ at the point $s \in [a, b]$ is defined as the solution of (H) with the initial conditions $\hat{X}_s(s) = 0$ and $\hat{U}_s(s) = I$, see also (3.1). We stress that the focal points are always counted including their multiplicities.

Proposition 1.2. *Assume (1.1) and let $m \in \mathbb{N} \cup \{0\}$ be fixed. The principal solution $(\hat{X}_a, \hat{U}_a)$ of (H) has m left proper focal points in $(a, b]$ if and only if the principal solution $(\hat{X}_b, \hat{U}_b)$ of (H) has m right proper focal points in $[a, b)$.*

Proposition 1.3. *Assume that (1.1) holds. If the principal solution $(\hat{X}_a, \hat{U}_a)$ of (H) has m left proper focal points in $(a, b]$, then any other conjoined basis (X, U) of (H) has at least m and at most $m + n$ left proper focal points in $(a, b]$. Similarly, if the principal solution $(\hat{X}_b, \hat{U}_b)$ of (H) has m right proper focal points in $[a, b)$, then any other conjoined basis (X, U) of (H) has at least m and at most $m + n$ right proper focal points in $[a, b)$.*

Proposition 1.4. *Assume that (1.1) holds. Then the difference between the numbers of left proper focal points in $(a, b]$ of any two conjoined bases of (H) is at most n. Similarly, the difference between the numbers of right proper focal points in $[a, b)$ of any two conjoined bases of (H) is at most n.*

We note that the statements in Propositions 1.3 and 1.4 regarding the right proper focal points follow from the corresponding results for the left proper focal points by the time-reversing transformation $t \mapsto a + b - t$, which is described in [24, Remark 4.7].

Linear Hamiltonian systems (H) without the complete controllability assumption are intensively studied in the literature. As we mentioned above, the second author derived in [34,35] general Sturmian separation and comparison theorems for such systems (H). Recently, Johnson, Novo, Nũnez, and Obaya proved in [20, Theorem 3.6] a very nice formula connecting the rotation number of system (H) with the number of left proper focal points of (X, U) in $(a, b]$ when $b \to \infty$. Uncontrollable systems (H) were also considered in [18,19,21] in the relation with the notion of a weak disconjugacy of (H) and dissipative control processes, and in [25,28–33] when studying the principal solutions of (H) at infinity.

In the present paper we derive (Theorem 4.1) new explicit formulas for the difference of the left proper focal points of two conjoined bases of (H) in $(a, b]$, and the right proper focal points in $[a, b)$, as well as optimal bounds for the numbers of left and right proper focal points of one conjoined basis in a bounded interval (Theorems 5.2, 5.3, 5.6, and 5.9 and Corollaries 5.8 and 5.10). These estimates essentially improve the statements in Propositions 1.3 and 1.4. In addition, we establish (Theorem 5.1) an exact relationship between the numbers of left proper focal points in $(a, b]$ and right proper focal points in $[a, b)$ of one conjoined basis of (H), which generalizes the formula in (1.2) to abnormal systems. We note that these results are new even in the case of a completely controllable system (H).

As a main tool in the above study we use the comparative index, which was introduced by Elyseeva in [12,13] and successfully applied in the *discrete* oscillation theory, see [6–10,14,16]. In this paper we study the properties of the comparative index from a point of view of the *continuous time* linear Hamiltonian system (H). In particular, we obtain the continuity and limit properties of the comparative index (Theorems 6.1 and 6.5) and their relationship with the multiplicities of left and right proper focal points at a given point (Theorem 6.3). In a sense, some results in this paper can be regarded as continuous time analogs of the known results for symplectic difference systems in [13], see Remark 7.4. This paper therefore provides a key step in the implementation of recent discrete time methods into the new continuous time theory.

The paper is organized as follows. In Section 2 we define the comparative index and display its algebraic properties, which are needed in this paper. In Sections 3 we recall some general facts about linear Hamiltonian systems and prove two auxiliary results about conjoined bases of (H). In Section 4 we derive key equalities relating the difference of the numbers proper focal points of two conjoined bases of (H) and the comparative index. In Sections 5 and 6 we establish further Sturmian separation theorems for conjoined bases of system (H) and the continuity and limit properties of the comparative index. Finally, in Section 7 we provide several examples illustrating our new theory, as well as comments about the related topics and future research directions.

2. Algebraic properties of comparative index

In this section we present the definition of the comparative index of two matrices and its main properties from [12,13]. Let Y and $\tilde{Y}$ be real constant $2n \times n$ matrices such that

$$Y^T \mathcal{J} Y = 0, \quad \tilde{Y}^T \mathcal{J} \tilde{Y} = 0, \quad \operatorname{rank} Y = n = \operatorname{rank} \tilde{Y}, \quad W := Y^T \mathcal{J} \tilde{Y}, \quad \mathcal{J} := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{2.1}$$

The matrix $\mathcal{J}$ is the canonical skew-symmetric matrix of dimension $2n$. The first three conditions in (2.1) can be regarded as suitable initial conditions for conjoined bases of system (H), while the matrix W is the Wronskian of Y and $\tilde{Y}$. When we split Y and $\tilde{Y}$ into $n \times n$ blocks $Y = (X^T, U^T)^T$ and $\tilde{Y} = (\tilde{X}^T, \tilde{U}^T)^T$, then the first, second, and fourth expressions in (2.1) have the form

$$X^T U = U^T X, \quad \tilde{X}^T \tilde{U} = \tilde{U}^T \tilde{X}, \quad W = X^T \tilde{U} - U^T \tilde{X}. \tag{2.2}$$

When considering the matrices Y and $\tilde{Y}$ satisfying (2.1), we will always express them in the above block structure with $X, U, \tilde{X}, \tilde{U}$ as in (2.2).

Following [12, Definition 2.1] or [13, Definition 2.1], we define the *comparative index* $\mu(Y, \tilde{Y})$ and the *dual comparative index* $\mu^*(Y, \tilde{Y})$ of Y and $\tilde{Y}$ as the numbers

$$\mu(Y, \tilde{Y}) := \operatorname{rank} \mathcal{M} + \operatorname{ind} \mathcal{P}, \quad \mu^*(Y, \tilde{Y}) := \operatorname{rank} \mathcal{M} + \operatorname{ind}(-\mathcal{P}), \tag{2.3}$$

where $\mathcal{M}$ and $\mathcal{P}$ are the $n \times n$ matrices

$$\mathcal{M} := (I - X^\dagger X)\, W, \quad \mathcal{P} := V W^T X^\dagger \tilde{X} V, \quad V := I - \mathcal{M}^\dagger \mathcal{M}, \tag{2.4}$$

and where W is the Wronskian of Y and $\tilde{Y}$ defined in (2.1). The dagger in (2.4) denotes the Moore–Penrose pseudoinverse of a matrix, i.e., the unique matrix $X^{\dagger}$ satisfying the four properties $X^{\dagger}X$ and $XX^{\dagger}$ are symmetric, $XX^{\dagger}X = X$, and $X^{\dagger}XX^{\dagger} = X^{\dagger}$. We refer to [1,2,4] for general theory of pseudoinverse matrices. We note that the matrix V in (2.4) is the orthogonal projector onto $\operatorname{Ker}\mathcal{M}$ and that the matrix $\mathcal{P}$ is symmetric, see [13, Theorem 2.1]. In this case $\operatorname{ind}\mathcal{P}$ denotes the number of negative eigenvalues of $\mathcal{P}$, and obviously we have $\operatorname{ind}(-\mathcal{P}) = \operatorname{rank}\mathcal{P} - \operatorname{ind}\mathcal{P}$.

For convenience we define the elementary $2n \times n$ matrix

$$E := (0,\ I)^T, \tag{2.5}$$

where 0 is the $n \times n$ zero matrix and I is the $n \times n$ identity matrix. The following properties in Propositions 2.1 and 2.2 are proven in [13, Section 2].

Proposition 2.1. *Let Y and $\tilde{Y}$ be $2n \times n$ matrices satisfying (2.1) and let E be given by (2.5). Then the comparative index $\mu(Y, \tilde{Y})$ and the dual comparative index $\mu^*(Y, \tilde{Y})$ defined in (2.3) have the following properties:*

$$\mu(Y, \tilde{Y}) + \operatorname{rank} X = \mu^*(\tilde{Y}, Y) + \operatorname{rank} \tilde{X}, \tag{2.6}$$

$$\mu(Y, \tilde{Y}) + \mu(\tilde{Y}, Y) = \operatorname{rank} W = \mu^*(Y, \tilde{Y}) + \mu^*(\tilde{Y}, Y), \tag{2.7}$$

$$\mu(Y, \tilde{Y}) \le \min\{\operatorname{rank} W,\ \operatorname{rank} \tilde{X}\}, \quad \mu^*(Y, \tilde{Y}) \le \min\{\operatorname{rank} W,\ \operatorname{rank} \tilde{X}\}, \tag{2.8}$$

$$\mu(Y, E) = 0 = \mu^*(Y, E), \quad \mu(E, Y) = \operatorname{rank} X = \mu^*(E, Y). \tag{2.9}$$

We note that the second conditions in (2.7) and (2.8) about the dual comparative index are not explicitly stated in [13], but they follow from (2.6) and from the corresponding properties for $\mu(Y, \tilde{Y})$ in (2.7) and (2.8).

We recall that a real $2n \times 2n$ matrix S is symplectic if $S^T \mathcal{J} S = \mathcal{J}$. Symplectic matrices are of basic importance for the theory of linear Hamiltonian system (H), since any fundamental matrix of (H) is symplectic on $[a, b]$ whenever it is symplectic at some initial point.

Proposition 2.2. *Let Z, $\tilde{Z}$, Φ be real $2n \times 2n$ symplectic matrices and let E be defined by (2.5). Then the following transformation formulas hold:*

$$\mu(\Phi Z E, \Phi \tilde{Z} E) - \mu(Z E, \tilde{Z} E) = \mu(\Phi Z E, \Phi E) - \mu(\Phi \tilde{Z} E, \Phi E), \tag{2.10}$$

$$\mu^*(\Phi Z E, \Phi \tilde{Z} E) - \mu^*(Z E, \tilde{Z} E) = \mu^*(\Phi Z E, \Phi E) - \mu^*(\Phi \tilde{Z} E, \Phi E). \tag{2.11}$$

Further properties of the comparative index and the dual comparative index are proven in [13]. In this paper we will also utilize the following new lower bounds of the comparative index, which complement the upper bounds in (2.8).

Lemma 2.3. *Let Y and $\tilde{Y}$ be $2n \times n$ matrices satisfying (2.1). Then*

$$0 \le \operatorname{rank} \tilde{X} - \min\{\operatorname{rank} \tilde{X},\ \operatorname{rank} X\} \le \min\{\mu(Y, \tilde{Y}),\ \mu^*(Y, \tilde{Y})\}, \tag{2.12}$$

$$0 \le \operatorname{rank} W - \min\{\operatorname{rank} W,\ \operatorname{rank} X\} \le \min\{\mu(Y, \tilde{Y}),\ \mu^*(Y, \tilde{Y})\}. \tag{2.13}$$

Proof. Let us put $p := \operatorname{rank} \tilde{X} - \min\{\operatorname{rank} X, \operatorname{rank} \tilde{X}\}$ and $q := \operatorname{rank} W - \min\{\operatorname{rank} X, \operatorname{rank} W\}$. Then $p \geq 0$ and $q \geq 0$. We know from (2.6) and $\mu^*(\tilde{Y}, Y) \geq 0$ that $\mu(Y, \tilde{Y}) \geq \operatorname{rank} \tilde{X} - \operatorname{rank} X$ holds, and similarly from (2.6) and $\mu(\tilde{Y}, Y) \geq 0$ we get $\mu^*(Y, \tilde{Y}) \geq \operatorname{rank} \tilde{X} - \operatorname{rank} X$. We will analyze the latter difference. If $\operatorname{rank} \tilde{X} \geq \operatorname{rank} X$, then $p = \operatorname{rank} \tilde{X} - \operatorname{rank} X$, while if $\operatorname{rank} \tilde{X} \leq \operatorname{rank} X$, then $p = 0$. But since $\mu(Y, \tilde{Y}) \geq 0$ and $\mu^*(Y, \tilde{Y}) \geq 0$, it follows from the above that $\mu(Y, \tilde{Y}) \geq p$ and $\mu^*(Y, \tilde{Y}) \geq p$, i.e., (2.12) holds. Next, the first conditions in (2.7) and (2.8) imply that

$$\mu(Y, \tilde{Y}) \overset{(2.7)}{=} \operatorname{rank} W - \mu(\tilde{Y}, Y) \overset{(2.8)}{\geq} \operatorname{rank} W - \min\{\operatorname{rank} W, \operatorname{rank} X\} = q.$$

In a similar way we obtain from the second conditions in (2.7) and (2.8) that $\mu^*(Y, \tilde{Y}) \geq q$. Therefore, the estimates in (2.13) hold as well. □

Applications of the comparative index in the oscillation theory of discrete symplectic systems can be found in [6–10,14,16]. In Section 4 below we will show how the comparative index arises in the theory of continuous time linear Hamiltonian system (H) and how it can be utilized in order to derive new Sturmian separation theorems for conjoined bases of (H).

3. Properties of linear Hamiltonian systems

A $2n \times n$ solution (X, U) of (H) is called a *conjoined basis* if $X^T(t_0)\, U(t_0)$ is symmetric and $\operatorname{rank}(X^T(t_0),\ U^T(t_0)) = n$ for some, and hence for any, point $t_0 \in [a, b]$, compare with (2.1). As an example of a conjoined basis of (H) we mention the *principal solution at the point* $s \in [a, b]$, denoted by $(\hat{X}_s, \hat{U}_s)$, which is defined as the solution of (H) with the initial conditions

$$\hat{X}_s(s) = 0, \quad \hat{U}_s(s) = I. \tag{3.1}$$

By [22, Corollary 3.3.9], any given conjoined basis (X, U) can be completed by another conjoined basis $(\bar{X}, \bar{U})$ to a (symplectic) fundamental matrix of (H), i.e., to a so-called *normalized* pair of conjoined bases, which then satisfy the relation $W := X^T(t)\, \bar{U}(t) - U^T(t)\, \bar{X}(t) = I$ on $[a, b]$, compare with (2.2). The fact that the fundamental matrix $\Phi := \begin{pmatrix} X & \bar{X} \\ U & \bar{U} \end{pmatrix}$ of (H) is symplectic is also equivalent with

$$X\bar{U}^T - \bar{X}U^T = I, \quad X\bar{X}^T = \bar{X}X^T, \quad U\bar{U}^T = \bar{U}U^T \tag{3.2}$$

on $[a, b]$, see [22, Proposition 1.1.5]. Moreover, similarly to [29–32] we say that (X, U) has constant kernel on the open or half-open interval (a, b), $(a, b]$, $[a, b)$, if the kernel of $X(t)$ is constant on that interval.

Let us fix for a moment a conjoined basis (X, U) of (H) with constant kernel on (a, b) and a point $\alpha \in (a, b)$. We define the constant orthogonal projector P onto $\operatorname{Im} X^T(t) = [\operatorname{Ker} X(t)]^{\perp}$, the symmetric $n \times n$ matrix function $S(t)$, and the orthogonal projector $P_{\mathcal{S}}(t)$ onto $\operatorname{Im} S(t)$ by

$$P \equiv P(t) := X^{\dagger}(t)\, X(t), \quad t \in (a, b), \tag{3.3}$$

$$S(t) := \int_{\alpha}^{t} X^{\dagger}(s)\, B(s)\, X^{\dagger\, T}(s)\, ds, \quad t \in (a, b), \tag{3.4}$$

$$P_{\mathcal{S}}(t) := S^{\dagger}(t)\, S(t) = S(t)\, S^{\dagger}(t), \quad t \in (a, b). \tag{3.5}$$

We note that $S(t)$ is correctly defined on (a, b), since in the present setting the function $X^{\dagger}(t)$ is piecewise continuously differentiable on (a, b), hence continuous, see [4, Theorems 10.5.1 and 10.5.3]. The next result follows from standard properties of symmetric monotone matrix-valued functions.

Lemma 3.1. *Assume that (1.1) holds, (X, U) is a conjoined basis of (H) with constant kernel on (a, b), and $\alpha \in (a, b)$ is given. Then the following hold.*

(i) *The matrix $S(t) \le 0$ on $(a, \alpha]$ and $S(t) \ge 0$ on $[\alpha, b)$.*
(ii) *The matrix $S(t)$ is nondecreasing and piecewise continuously differentiable on (a, b) with $S'(t) = X^{\dagger}(t)\, B(t)\, X^{\dagger\, T}(t) \ge 0$ on (a, b).*
(iii) *The set $\operatorname{Im} S(t)$ is nonincreasing on $(a, \alpha]$ and nondecreasing on $[\alpha, b)$ with $\operatorname{Im} S(t) = \operatorname{Im} P_{\mathcal{S}}(t) \subseteq \operatorname{Im} P$ on (a, b), and the following limits exist*

$$P_{\mathcal{S}a} := \lim_{t \to a^+} P_{\mathcal{S}}(t), \quad \operatorname{Im} P_{\mathcal{S}a} \subseteq \operatorname{Im} P, \tag{3.6}$$

$$P_{\mathcal{S}b} := \lim_{t \to b^-} P_{\mathcal{S}}(t), \quad \operatorname{Im} P_{\mathcal{S}b} \subseteq \operatorname{Im} P. \tag{3.7}$$

(iv) *The matrix $S^{\dagger}(t)$ is nonincreasing on $(a, a+\varepsilon)$ and on $(b-\varepsilon, b)$ for some $\varepsilon \in (0, b-a)$ and the following limits exist*

$$T_a := \lim_{t \to a^+} S^{\dagger}(t), \quad T_a \le 0, \quad \operatorname{Im} T_a \subseteq \operatorname{Im} P_{\mathcal{S}a}, \tag{3.8}$$

$$T_b := \lim_{t \to b^-} S^{\dagger}(t), \quad T_b \ge 0, \quad \operatorname{Im} T_b \subseteq \operatorname{Im} P_{\mathcal{S}b}. \tag{3.9}$$

Proof. The statements in parts (i) and (ii) follow by direct considerations from the definition of $S(t)$ in (3.4). The monotonicity property of $\operatorname{Im} S(t)$ in part (iii) is proven in [29, Theorem 4.2], which then yields that the set $\operatorname{Im} S(t)$ and hence the matrix $P_{\mathcal{S}}(t)$ are constant in some right neighborhood of a and in some left neighborhood of b. This shows that the limits in (3.6)–(3.7) exist. Part (iv) follows from the fact that the image of $S(t)$ is constant on some right neighborhood $(a, a+\varepsilon)$ of a and on some left neighborhood $(b-\varepsilon, b)$ of b by part (iii), and $S^{\dagger}(t)$ is nonincreasing on these neighborhoods by part (ii). Therefore, the limits in (3.8)–(3.9) indeed exist and have the stated properties. □

We note that the matrices $P_{\mathcal{S}a}$ and $P_{\mathcal{S}b}$ in (3.6)–(3.7) are the maximal orthogonal projectors onto $\operatorname{Im} S(t)$ on $(a, \alpha]$ and on $[\alpha, b)$, respectively. Next we relate the matrix $S(t)$ with some special conjoined basis $(\bar{X}, \bar{U})$ of (H) associated with (X, U) and $\alpha \in (a, b)$.

Lemma 3.2. *Assume that (X, U) is a conjoined basis of (H) with constant kernel on (a, b) and $\alpha \in (a, b)$ is given. Let the matrices P, $S(t)$, $P_{\mathcal{S}}(t)$ and T_a, T_b be defined by (3.3)–(3.5) and (3.8)–(3.9). Then there exists a conjoined basis $(\bar{X}, \bar{U})$ of (H) such that (X, U) and $(\bar{X}, \bar{U})$ are normalized and*

$$X^{\dagger}(\alpha)\, \bar{X}(\alpha) = 0. \tag{3.10}$$

Moreover, the matrix $\bar{X}(t)$ also satisfies for all $t \in (a, b)$ the identities

$$S(t) = X^{\dagger}(t)\, \bar{X}(t)\, P, \quad \bar{X}(t)\, P = X(t)\, S(t), \tag{3.11}$$

$$\bar{P}(t) := \bar{X}^{\dagger}(t)\, \bar{X}(t) = I - P + P_{\mathcal{S}}(t), \tag{3.12}$$

$$S^{\dagger}(t) = \bar{X}^{\dagger}(t)\, X(t)\, \bar{P}(t) = \bar{X}^{\dagger}(t)\, X(t)\, P_{\mathcal{S}}(t), \tag{3.13}$$

$$\bar{X}(t)\, S^{\dagger}(t)\, \bar{X}^{T}(t) = X(t)\, \bar{X}^{T}(t). \tag{3.14}$$

If in addition condition (1.1) holds, then

$$X(a)\, \bar{X}^{T}(a) = \bar{X}(a)\, T_a\, \bar{X}^{T}(a), \quad X(b)\, \bar{X}^{T}(b) = \bar{X}(b)\, T_b\, \bar{X}^{T}(b), \tag{3.15}$$

$$X(t)\, \bar{X}^{T}(t) \leq 0 \text{ on } [a, \alpha], \quad X(t)\, \bar{X}^{T}(t) \geq 0 \text{ on } [\alpha, b]. \tag{3.16}$$

Proof. The existence of a conjoined basis $(\bar{X}, \bar{U})$, which is normalized with (X, U) and which satisfies (3.10) and (3.11) is proven in [29, Theorem 4.4 and Remark 4.5(ii)]. We note that condition (1.1) is not needed in the assumption of the latter reference. By [28, Theorem 2.2.11(i)] we know that $\operatorname{Ker} \bar{X}(t) = \operatorname{Im}[P - P_{\mathcal{S}}(t)]$ on (a, b), which means that the matrix $I - P + P_{\mathcal{S}}(t)$ is the orthogonal projector onto the set $\operatorname{Im} \bar{X}^{T}(t)$, i.e., identity (3.12) holds. Next, by the identities (suppressing the argument t) $X = XP$, $P P_{\mathcal{S}} = P_{\mathcal{S}} = P_{\mathcal{S}} P$, and $P_{\mathcal{S}} S^{\dagger} = S^{\dagger}$ we obtain

$$\begin{aligned} \bar{X}^{\dagger} X \bar{P} &\overset{(3.12)}{=} \bar{X}^{\dagger} X P\, (I - P + P_{\mathcal{S}}) = \bar{X}^{\dagger} X P_{\mathcal{S}} \overset{(3.5)}{=} \bar{X}^{\dagger} X S S^{\dagger} \overset{(3.11)}{=} \bar{X}^{\dagger} \bar{X} P S^{\dagger} \\ &\overset{(3.12)}{=} \bar{P} P S^{\dagger} = (I - P + P_{\mathcal{S}})\, P S^{\dagger} = P_{\mathcal{S}} P S^{\dagger} = P_{\mathcal{S}} S^{\dagger} = S^{\dagger}, \end{aligned}$$

which proves the identities in (3.13). In turn, we have by the symmetry of $X \bar{X}^{T}$, see (3.2),

$$\bar{X} S^{\dagger} \bar{X}^{T} \overset{(3.13)}{=} \bar{X} \bar{X}^{\dagger} X \bar{P} \bar{X}^{T} = \bar{X} \bar{X}^{\dagger} X \bar{X}^{T} = \bar{X} \bar{X}^{\dagger} \bar{X} X^{T} = \bar{X} X^{T} = X \bar{X}^{T},$$

which shows that identity (3.14) also holds. Upon taking the limit as $t \to a^{+}$ and $t \to b^{-}$ in (3.14) and using the definition of T_a, T_b in (3.8)–(3.9) and the continuity of $X(t)$, $\bar{X}(t)$ we obtain (3.15). Finally, by Lemma 3.1(i) we know that $S^{\dagger}(t) \leq 0$ on $(a, \alpha]$ and $S^{\dagger}(t) \geq 0$ on $[\alpha, b)$, since a matrix and its pseudoinverse have the same definiteness properties. Therefore, the inequalities in (3.16) follow from (3.14) and (3.15). The proof is complete. □

Remark 3.3. It is obvious to see that if the conjoined basis (X, U) has constant kernel on $(a, b]$, then the statements in Lemma 3.2 hold also with $\alpha = b$. Similarly, if (X, U) has constant kernel on $[a, b)$, then the statements in Lemma 3.2 hold also with $\alpha = a$.

For completeness we note that the conjoined basis $(\bar{X}, \bar{U})$ in Lemma 3.2 is not uniquely determined by (X, U) in the function $\bar{U}$. On the other hand, the solution $(\bar{X} P, \bar{U} P)$ of (H) is uniquely determined by (X, U), as we showed in [29, Remark 4.5(ii)].

Remark 3.4. In order to measure to what extent the controllability assumption on system (H) is violated, the following quantity was introduced in [25, Section 3], see also [29, Section 5] or [31, Section 3]. For fixed $\alpha, \beta \in [a, b]$ with $\alpha < \beta$ we denote by $\Lambda[\alpha, \beta]$ the linear space of piecewise continuously differentiable vector-valued functions $u : [\alpha, \beta] \to \mathbb{R}^{n}$, which satisfy the equations $u' = -A^{T}(t)\, u$ and $B(t)\, u = 0$ on $[\alpha, \beta]$. The functions $u \in \Lambda[\alpha, \beta]$ correspond to the solutions $(x \equiv 0, u)$ of system (H) on $[\alpha, \beta]$. The space $\Lambda[\alpha, \beta]$ is finite-dimensional with $d[\alpha, \beta] := \dim \Lambda[\alpha, \beta] \le n$. The number $d[\alpha, \beta]$ is called the *order of abnormality* of system (H) on the interval $[\alpha, \beta]$. We remark that system (H) is called *normal* on $[\alpha, \beta]$ if $d[\alpha, \beta] = 0$, while it is called *identically normal* (or *completely controllable*) on $[\alpha, \beta]$ if $d(J) = 0$ for every nondegenerate subinterval $J \subseteq [\alpha, \beta]$. The integer-valued function $d[t, \beta]$ is nondecreasing, piecewise constant, and right-continuous on $[a, \beta]$. Similarly, the integer-valued function $d[\alpha, t]$ is nonincreasing, piecewise constant, and left-continuous on $[\alpha, b]$. This implies that the limits

$$d_\alpha^+ := \lim_{t \to \alpha^+} d[\alpha, t] = \max_{t \in (\alpha, b]} d[\alpha, t], \quad 0 \le d_\alpha^+ \le n, \tag{3.17}$$

$$d_\beta^- := \lim_{t \to \beta^-} d[t, \beta] = \max_{t \in [a, \beta)} d[t, \beta], \quad 0 \le d_\beta^- \le n, \tag{3.18}$$

exist. The numbers d_α^+ and d_β^- are called the *maximal orders of abnormality of* (H) *at the point* α *from the right* and *at the point* β *from the left*, respectively. Since $d[\alpha, t]$ and $d[t, \beta]$ are integer-valued functions, the existence of the limits in (3.17) and (3.18) implies that

$$d[\alpha, t] \equiv d_\alpha^+ \text{ for all } t \in (\alpha, \alpha + \varepsilon], \text{ and } d[t, \beta] \equiv d_\beta^- \text{ for all } t \in [\beta - \varepsilon, \beta)$$

for some $\varepsilon > 0$. It is known in [28, Theorem 3.1.2], see also [29, Theorem 5.2], that these constant quantities are related to the rank of the principal solutions $(\hat{X}_\alpha, \hat{U}_\alpha)$ and $(\hat{X}_\beta, \hat{U}_\beta)$. More precisely, $\operatorname{Ker} \hat{X}_\alpha(t)$ is constant on $(\alpha, \alpha + \varepsilon]$, $\operatorname{Ker} \hat{X}_\beta(t)$ is constant on $[\beta - \varepsilon, \beta)$, and

$$\operatorname{rank} \hat{X}_\alpha(t) = n - d[\alpha, t] \equiv n - d_\alpha^+ \quad \text{for all } t \in (\alpha, \alpha + \varepsilon], \tag{3.19}$$

$$\operatorname{rank} \hat{X}_\beta(t) = n - d[t, \beta] \equiv n - d_\beta^- \quad \text{for all } t \in [\beta - \varepsilon, \beta). \tag{3.20}$$

The relations in (3.19) and (3.20) will be utilized in Section 6 when studying the limit properties of the comparative index.

4. Comparative index and continuous Sturmian theory

In this section we derive two main equalities involving the comparative index, as defined in (2.3), in the context of the continuous time linear Hamiltonian system (H). We recall the notation $(\hat{X}_a, \hat{U}_a)$ and $(\hat{X}_b, \hat{U}_b)$ introduced in (3.1) for the principal solutions of (H) at the points a and b, respectively. Moreover, if (X, U) and $(\tilde{X}, \tilde{U})$ are conjoined basis of (H), then we set $Y := (X^T, U^T)^T$ and $\tilde{Y} := (\tilde{X}^T, \tilde{U}^T)^T$, that is, we define for $t \in [a, b]$ the $2n \times n$ matrices

$$Y(t) := \begin{pmatrix} X(t) \\ U(t) \end{pmatrix}, \quad \tilde{Y}(t) := \begin{pmatrix} \tilde{X}(t) \\ \tilde{U}(t) \end{pmatrix}, \quad \hat{Y}_a(t) := \begin{pmatrix} \hat{X}_a(t) \\ \hat{U}_a(t) \end{pmatrix}, \quad \hat{Y}_b(t) := \begin{pmatrix} \hat{X}_b(t) \\ \hat{U}_b(t) \end{pmatrix}. \tag{4.1}$$

The central result of this section allows to express the difference between the numbers of left proper focal points in $(a, b]$ for two conjoined bases (X, U) and $(\tilde{X}, \tilde{U})$ of (H) in terms of the comparative index evaluated at the endpoints of the considered interval. Similarly for the right proper focal points of (X, U) and $(\tilde{X}, \tilde{U})$ in $[a, b)$ we use the dual comparative index. For this purpose we introduce the notation

$$m_L(a,b] := \text{ the number of left proper focal points of } (X,U) \text{ in } (a,b], \tag{4.2}$$

$$m_R[a,b) := \text{ the number of right proper focal points of } (X,U) \text{ in } [a,b), \tag{4.3}$$

$$\widetilde{m}_L(a,b] := \text{ the number of left proper focal points of } (\tilde{X},\tilde{U}) \text{ in } (a,b], \tag{4.4}$$

$$\widetilde{m}_R[a,b) := \text{ the number of right proper focal points of } (\tilde{X},\tilde{U}) \text{ in } [a,b). \tag{4.5}$$

The left and right proper focal points are always counted including their multiplicities. By (1.5) and (1.6) we then have the equalities

$$m_L(a,b] = \sum_{t\in(a,b]} m_L(t), \quad m_R[a,b) = \sum_{t\in[a,b)} m_R(t).$$

We note that under (1.1) the above sums are always finite, since the kernel of $X(t)$ is piecewise constant on $[a,b]$ by Proposition 1.1. With the notation in (4.2)–(4.5) we prove the following main result, which implements the comparative index into the Sturmian theory of continuous time linear Hamiltonian systems (H).

Theorem 4.1 *(Sturmian separation theorem). Assume that (1.1) holds. Then for any conjoined bases (X,U) and $(\tilde{X},\tilde{U})$ of (H) we have the equalities*

$$m_L(a,b] - \widetilde{m}_L(a,b] = \mu\big(Y(b),\tilde{Y}(b)\big) - \mu\big(Y(a),\tilde{Y}(a)\big), \tag{4.6}$$

$$m_R[a,b) - \widetilde{m}_R[a,b) = \mu^*\big(Y(a),\tilde{Y}(a)\big) - \mu^*\big(Y(b),\tilde{Y}(b)\big). \tag{4.7}$$

The proof of Theorem 4.1 is presented at the end of this section. The idea is to prove equalities (4.6) and (4.7) first under an additional assumption on the constant kernel of (X,U) and $(\tilde{X},\tilde{U})$ on $(a,b]$ or $[a,b)$, and then apply this partial statement to a suitable partition of the interval $[a,b]$. Before we proceed to this step we derive several auxiliary statements about the comparative index.

Proposition 4.2. *Let (X,U) and $(\tilde{X},\tilde{U})$ be conjoined bases of (H). Then for all $t\in[a,b]$*

$$\mu\big(Y(t),\tilde{Y}(t)\big) - \mu\big(Y(a),\tilde{Y}(a)\big) = \mu\big(Y(t),\hat{Y}_a(t)\big) - \mu\big(\tilde{Y}(t),\hat{Y}_a(t)\big), \tag{4.8}$$

$$\mu^*\big(Y(t),\tilde{Y}(t)\big) - \mu^*\big(Y(b),\tilde{Y}(b)\big) = \mu^*\big(Y(t),\hat{Y}_b(t)\big) - \mu^*\big(\tilde{Y}(t),\hat{Y}_b(t)\big). \tag{4.9}$$

Proof. Let E be the matrix in (2.5), so that $\hat{Y}_a(a) = E$. Let $\Phi(t)$, $\widetilde{\Phi}(t)$, $\hat{\Phi}_a(t)$, $\hat{\Phi}_b(t)$ be the fundamental matrices of system (H) such that

$$Y(t) = \Phi(t)\,E, \quad \tilde{Y}(t) = \widetilde{\Phi}(t)\,E, \quad \hat{\Phi}_a(t)\,E = \hat{Y}_a(t), \quad \hat{\Phi}_b(t)\,E = \hat{Y}_b(t), \quad t\in[a,b]. \tag{4.10}$$

That is, the fundamental matrices $\Phi(t)$, $\widetilde{\Phi}(t)$, $\hat{\Phi}_a(t)$, $\hat{\Phi}_b(t)$ are symplectic and they are constructed in such a way that the conjoined bases $Y(t)$, $\tilde{Y}(t)$, $\hat{Y}_a(t)$, $\hat{Y}_b(t)$ form their second $2n\times n$ matrix columns, respectively. This can be done by a suitable choice of conjoined bases which complete $Y(t)$, $\tilde{Y}(t)$, $\hat{Y}_a(t)$, $\hat{Y}_b(t)$ to normalized conjoined bases of (H). For example, the first $2n\times n$ column of the matrix $\Phi(t)$ can be given by $\bar{Y}(t) := (-\bar{X}^T(t), -\bar{U}^T(t))^T$, where $(\bar{X},\bar{U})$

is the conjoined basis from Lemma 3.2 associated with (X, U), or the first $2n \times n$ column of the matrix $\hat{\Phi}_a(t)$ is given by $\bar{Y}_a(t) := (\bar{X}_a^T(t), \bar{U}_a^T(t))^T$, where $(\bar{X}_a, \bar{U}_a)$ is the conjoined basis of (H) with the initial conditions $\bar{X}_a(a) = I$ and $\bar{U}_a(a) = 0$. In particular, we can construct $\hat{\Phi}_a(t)$ and $\hat{\Phi}_b(t)$ such that they satisfy $\hat{\Phi}_a(a) = I = \hat{\Phi}_b(b)$. It then follows by the uniqueness of solutions of system (H) that

$$\hat{\Phi}_a(t)\,\Phi(a) = \Phi(t) = \hat{\Phi}_b(t)\,\Phi(b), \quad \hat{\Phi}_a(t)\,\widetilde{\Phi}(a) = \widetilde{\Phi}(t) = \hat{\Phi}_b(t)\,\widetilde{\Phi}(b), \quad t \in [a, b]. \tag{4.11}$$

By using (4.10) and (4.11) and applying formula (2.10) in Proposition 2.2 with the symplectic matrices $\Phi := \hat{\Phi}_a(t)$, $Z := \Phi(a)$, and $\tilde{Z} := \widetilde{\Phi}(a)$, we then obtain the equality

$$\begin{aligned} \mu\big(Y(t), \tilde{Y}(t)\big) - \mu\big(Y(a), \tilde{Y}(a)\big) &= \mu\big(\hat{\Phi}_a(t)\,\Phi(a)\,E, \hat{\Phi}_a(t)\,\widetilde{\Phi}(a)\,E\big) - \mu\big(\Phi(a)\,E, \widetilde{\Phi}(a)\,E\big) \\ &\overset{(2.10)}{=} \mu\big(\hat{\Phi}_a(t)\,\Phi(a)\,E, \hat{\Phi}_a(t)\,E\big) - \mu\big(\hat{\Phi}_a(t)\,\widetilde{\Phi}(a)\,E, \hat{\Phi}_a(t)\,E\big) \\ &= \mu\big(Y(t), \hat{Y}_a(t)\big) - \mu\big(\tilde{Y}(t), \hat{Y}_a(t)\big). \end{aligned}$$

Therefore, (4.8) is proven. Similarly, by applying formula (2.11) in Proposition 2.2 with the symplectic matrices $\Phi := \hat{\Phi}_b(t)$, $Z := \Phi(b)$, and $\tilde{Z} := \widetilde{\Phi}(b)$, we obtain

$$\begin{aligned} \mu^*\big(Y(t), \tilde{Y}(t)\big) - \mu^*\big(Y(b), \tilde{Y}(b)\big) &= \mu^*\big(\hat{\Phi}_b(t)\,\Phi(b)\,E, \hat{\Phi}_b(t)\,\widetilde{\Phi}(b)\,E\big) \\ &\quad - \mu^*\big(\Phi(b)\,E, \widetilde{\Phi}(b)\,E\big) \\ &\overset{(2.11)}{=} \mu^*\big(\hat{\Phi}_b(t)\,\Phi(b)\,E, \hat{\Phi}_b(t)\,E\big) \\ &\quad - \mu^*\big(\hat{\Phi}_b(t)\,\widetilde{\Phi}(b)\,E, \hat{\Phi}_b(t)\,E\big) \\ &= \mu^*\big(Y(t), \hat{Y}_b(t)\big) - \mu^*\big(\tilde{Y}(t), \hat{Y}_b(t)\big). \end{aligned}$$

The proof of (4.9) is also complete. □

Proposition 4.3. *Let (X, U) be a conjoined basis of (H) with constant kernel on $(a, b]$. Then*

$$\mu\big(Y(b), \hat{Y}_a(b)\big) = 0, \quad \mu^*\big(Y(a), \hat{Y}_b(a)\big) = \operatorname{rank} X(b) - \operatorname{rank} X(a). \tag{4.12}$$

Proof. Let $P(t)$ be the orthogonal projector onto $\operatorname{Im} X^T(t)$ defined in (3.3). Since we now assume that the kernel of $X(t)$ is constant on $(a, b]$, it follows by the continuity that $\operatorname{Ker} X(t) \equiv \operatorname{Ker} X(a^+) \subseteq \operatorname{Ker} X(a)$ on $(a, b]$, so that $\operatorname{Im} P(a) \subseteq \operatorname{Im} P(b)$. This in turn implies that $P(b)\,P(a) = P(a) = P(a)\,P(b)$. By (2.3)–(2.4), we have $\mu\big(Y(b), \hat{Y}_a(b)\big) = \operatorname{rank} \mathcal{M} + \operatorname{ind} \mathcal{P}$, where $\mathcal{M} = [I - P(b)]\,W$, $V = I - \mathcal{M}^\dagger \mathcal{M}$, $\mathcal{P} = V\,W^T X^\dagger(b)\,\hat{X}_a(b)\,V$, and where the constant Wronskian $W = X^T(t)\,\hat{U}_a(t) - U^T(t)\,\hat{X}_a(t)$ on $[a, b]$. In particular, evaluating the Wronskian at $t = a$ yields that $W = X^T(a)$ and hence,

$$\left.\begin{aligned} \mathcal{M} = [I - P(b)]\,X^T(a) = [I - P(b)]\,P(a)\,X^T(a) = [I - P(b)]\,P(b)\,P(a)\,X^T(a) = 0, \\ V = I, \quad \mathcal{P} = X(a)\,X^\dagger(b)\,\hat{X}_a(b). \end{aligned}\right\} \tag{4.13}$$

By Remark 3.3 and Lemma 3.2 with $\alpha = b$ we have for $(\hat{X}_a, \hat{U}_a)$ the representation

$$\begin{pmatrix} \hat{X}_a(t) \\ \hat{U}_a(t) \end{pmatrix} = \begin{pmatrix} X(t) & \bar{X}(t) \\ U(t) & \bar{U}(t) \end{pmatrix} \begin{pmatrix} -\bar{X}^T(a) \\ X^T(a) \end{pmatrix} = \begin{pmatrix} -X(t)\,\bar{X}^T(a) + \bar{X}(t)\,X^T(a) \\ -U(t)\,\bar{X}^T(a) + \bar{U}(t)\,X^T(a) \end{pmatrix}, \quad t \in [a,b],$$

where $(\bar{X}, \bar{U})$ is a conjoined basis of (H) such that (X, U) and $(\bar{X}, \bar{U})$ are normalized and $X^\dagger(b)\,\bar{X}(b) = 0$ according to (3.10). From this we obtain

$$\begin{aligned} \mathcal{P} &\overset{(4.13)}{=} X(a)\,X^\dagger(b)\,\hat{X}_a(b) = X(a)\,X^\dagger(b)\,[-X(b)\,\bar{X}^T(a) + \bar{X}(b)\,X^T(a)] \\ &\overset{(3.10)}{=} -X(a)\,P(a)\,P(b)\,\bar{X}^T(a) = -X(a)\,P(a)\,\bar{X}^T(a) = -X(a)\,\bar{X}^T(a) \overset{(3.16)}{\geq} 0. \end{aligned} \tag{4.14}$$

The calculations in (4.13) and (4.14) show that $\mu\big(Y(b), \hat{Y}_a(b)\big) = \operatorname{rank}\mathcal{M} + \operatorname{ind}\mathcal{P} = 0$, which proves the first equality in (4.12).

Next we evaluate the dual comparative index $\mu^*\big(Y(a), \hat{Y}_b(a)\big) = \operatorname{rank}\mathcal{M} + \operatorname{ind}\mathcal{P}$, where $\mathcal{M} = [I - P(a)]\,W$, $V = I - \mathcal{M}^\dagger\mathcal{M}$, $\mathcal{P} = -V\,W^T X^\dagger(a)\,\hat{X}_b(a)\,V$, and where the constant Wronskian $W = X^T(t)\,\hat{U}_b(t) - U^T(t)\,\hat{X}_b(t)$ on $[a,b]$. Evaluating the Wronskian at $t = b$ yields that $W = X^T(b)$ and hence,

$$\left.\begin{aligned} \mathcal{M} = [I - P(a)]\,X^T(b), \quad \operatorname{Im}\mathcal{M} = \operatorname{Im}[I - P(a)]\,P(b) = \operatorname{Im}[P(b) - P(a)], \\ \operatorname{rank}\mathcal{M} = \operatorname{rank}P(b) - \operatorname{rank}P(a) = \operatorname{rank}X(b) - \operatorname{rank}X(a). \end{aligned}\right\} \tag{4.15}$$

Next we shall prove that

$$\operatorname{Im}X^T(b)\,V = \operatorname{Im}P(a), \qquad \text{i.e.,} \qquad X^T(b)\,V = P(a)\,K \tag{4.16}$$

for some invertible matrix K. Since $\mathcal{M}V = 0$, it follows that $P(a)\,X^T(b)\,V = X^T(b)\,V$ and hence, $\operatorname{Im}X^T(b)\,V \subseteq \operatorname{Im}P(a)$. Conversely, assume that $v \in \operatorname{Im}P(a)$. Then we also have $v \in \operatorname{Im}P(b) = \operatorname{Im}X^T(b)$ and there exists $w \in \mathbb{R}^n$ such that $v = X^T(b)\,w$. Then we write $X^T(b)\,Vw = X^T(b)\,w - X^T(b)\,\mathcal{M}^\dagger\mathcal{M}w = v - X^T(b)\,\mathcal{M}^\dagger\mathcal{M}w$. But by using (4.15) we have $\mathcal{M}w = [I - P(a)]\,X^T(b)\,w = [I - P(a)]\,v = 0$, so that $v = X^T(b)\,Vw$. Therefore, $v \in \operatorname{Im}X^T(b)\,V$ and (4.16) is proven.

By Remark 3.3 and Lemma 3.2 with $\alpha = b$ we have for $(\hat{X}_b, \hat{U}_b)$ the representation

$$\begin{pmatrix} \hat{X}_b(t) \\ \hat{U}_b(t) \end{pmatrix} = \begin{pmatrix} X(t) & \bar{X}(t) \\ U(t) & \bar{U}(t) \end{pmatrix} \begin{pmatrix} -\bar{X}^T(b) \\ X^T(b) \end{pmatrix} = \begin{pmatrix} -X(t)\,\bar{X}^T(b) + \bar{X}(t)\,X^T(b) \\ -U(t)\,\bar{X}^T(b) + \bar{U}(t)\,X^T(b) \end{pmatrix}, \quad t \in [a,b],$$

where $(\bar{X}, \bar{U})$ is a conjoined basis of (H) such that (X, U) and $(\bar{X}, \bar{U})$ are normalized and $X^\dagger(b)\,\bar{X}(b) = 0$ according to (3.10). This yields that $\hat{X}_b(a) = -X(a)\,\bar{X}^T(b) + \bar{X}(a)\,X^T(b)$. But

$$\begin{aligned} X(a)\,\bar{X}^T(b) &= X(a)\,P(a)\,P(b)\,\bar{X}^T(b) = X(a)\,X^\dagger(b)\,X(b)\,\bar{X}^T(b) \\ &\overset{(3.2)}{=} X(a)\,X^\dagger(b)\,\bar{X}(b)\,X^T(b) = 0, \end{aligned}$$

which implies that $\hat{X}_b(a) = \bar{X}(a)\,X^T(b)$. For the matrix $\mathcal{P}$ we then have by (4.16) the expression

$$\mathcal{P} = -V\, X(b)\, X^{\dagger}(a)\, \bar{X}(a)\, X^T(b)\, V \overset{(4.16)}{=} -K^T\, P(a)\, X^{\dagger}(a)\, \bar{X}(a)\, P(a)\, K$$
$$= -K^T\, X^{\dagger}(a)\, \bar{X}(a)\, P(a)\, K = -K^T\, X^{\dagger}(a)\, \bar{X}(a)\, X^T(a)\, X^{\dagger T}(a)\, K \overset{(3.16)}{\geq} 0.$$

Therefore, the equality $\operatorname{ind}\mathcal{P} = 0$ holds. From the definition of the dual comparative index and (4.15) we then obtain the formula $\mu^*\big(Y(a), \hat{Y}_b(a)\big) = \operatorname{rank}\mathcal{M} + \operatorname{ind}\mathcal{P} = \operatorname{rank} X(b) - \operatorname{rank} X(a)$, which proves the second equality in (4.12). □

Proposition 4.4. *Let (X, U) be a conjoined basis of (H) with constant kernel on $[a, b)$. Then*

$$\mu\big(Y(b), \hat{Y}_a(b)\big) = \operatorname{rank} X(a) - \operatorname{rank} X(b), \quad \mu^*\big(Y(a), \hat{Y}_b(a)\big) = 0. \tag{4.17}$$

Proof. The proof is similar to the proof of Proposition 4.4 with the difference that $\alpha = a$ is used in Remark 3.3 and Lemma 3.2 instead of $\alpha = b$. The details are therefore omitted. □

With the notation in (4.1), (4.2), and (4.3) we have the following.

Corollary 4.5. *Let (X, U) be a conjoined basis of (H) with constant kernel on the interval $(a, b]$ or on the interval $[a, b)$. Then*

$$m_L(a, b] = \mu\big(Y(b), \hat{Y}_a(b)\big), \quad m_R[a, b) = \mu^*\big(Y(a), \hat{Y}_b(a)\big). \tag{4.18}$$

Proof. If (X, U) has constant kernel on $(a, b]$, then $\operatorname{Ker} X(t) \equiv \operatorname{Ker} X(a^+) \subseteq \operatorname{Ker} X(a)$ for all $t \in (a, b]$. Therefore, $m_L(a, b] = 0 = \mu\big(Y(b), \hat{Y}_a(b)\big)$ and $m_R[a, b) = \operatorname{def} X(a) - \operatorname{def} X(b) = \operatorname{rank} X(b) - \operatorname{rank} X(a) = \mu^*\big(Y(a), \hat{Y}_b(a)\big)$, both by (4.12) in Proposition 4.3. Similarly, if (X, U) has constant kernel on $[a, b)$, then $\operatorname{Ker} X(t) \equiv \operatorname{Ker} X(b^-) \subseteq \operatorname{Ker} X(b)$ for all $t \in (a, b]$. In this case $m_L(a, b] = \operatorname{def} X(b) - \operatorname{def} X(a) = \operatorname{rank} X(a) - \operatorname{rank} X(b) = \mu\big(Y(b), \hat{Y}_a(b)\big)$ and also $m_R[a, b) = 0 = \mu^*\big(Y(a), \hat{Y}_b(a)\big)$, both by (4.17) in Proposition 4.4. Therefore, if (X, U) has constant kernel on $(a, b]$ or $[a, b)$, then in both cases the two formulas in (4.18) hold. □

Now we can proceed with the proofs of key equalities (4.6) and (4.7) in Theorem 4.1.

Proof of Theorem 4.1. Step I. First we prove that the equalities in (4.6)–(4.7) hold under the additional assumption that (X, U), resp. $(\tilde{X}, \tilde{U})$, has constant kernel on $(a, b]$ or on $[a, b)$. In this case we know by (4.18) applied to (X, U) and $(\tilde{X}, \tilde{U})$ that $m_L(a, b] = \mu\big(Y(b), \hat{Y}_a(b)\big)$ and $\widetilde{m}_L(a, b] = \mu\big(\tilde{Y}(b), \hat{Y}_a(b)\big)$. Upon subtracting these two terms and using identity (4.8) at $t = b$ we obtain equality (4.6). Similarly, by (4.18) we know that $m_R[a, b) = \mu^*\big(Y(a), \hat{Y}_b(a)\big)$ and $\widetilde{m}_R[a, b) = \mu^*\big(\tilde{Y}(a), \hat{Y}_b(a)\big)$, which in combination with identity (4.9) at $t = a$ yields (4.7).

Step II. Let now (X, U) and $(\tilde{X}, \tilde{U})$ be arbitrary two conjoined bases of (H). From (1.1) it follows through Proposition 1.1 that (X, U) and $(\tilde{X}, \tilde{U})$ have piecewise constant kernel on $[a, b]$, that is, there exists a common partition $\mathcal{T} := \{a = t_0 < t_1 < \cdots < t_k = b\}$ of the interval $[a, b]$ such that for every $j \in \{0, 1, \dots, k-1\}$ the kernel of $X(t)$, resp. the kernel of $\tilde{X}(t)$, is constant on $(t_j, t_{j+1}]$ or on $[t_j, t_{j+1})$. Indeed, if $a = \tau_0 < \tau_1 < \cdots < \tau_m = b$ are the points where the kernel of $X(t)$ or the kernel of $\tilde{X}(t)$ changes, then the desired partition $\mathcal{T}$ is formed by the points τ_i, $i \in \{0, 1, \dots, m\}$, and e.g. by the midpoints of the intervals (τ_i, τ_{i+1}) for $i \in \{0, 1, \dots, m-1\}$. By Part I of the proof we may apply identities (4.6) and (4.7) on each partition subinterval $[t_j, t_{j+1}]$ to get

$$m_L(t_j, t_{j+1}] - \widetilde{m}_L(t_j, t_{j+1}] = \mu\big(Y(t_{j+1}), \tilde{Y}(t_{j+1})\big) - \mu\big(Y(t_j), \tilde{Y}(t_j)\big),$$
$$m_R[t_j, t_{j+1}) - \widetilde{m}_R[t_j, t_{j+1}) = \mu^*\big(Y(t_j), \tilde{Y}(t_j)\big) - \mu^*\big(Y(t_{j+1}), \tilde{Y}(t_{j+1})\big)$$

for every $j \in \{0, 1, \dots, k-1\}$. The results in (4.6) and (4.7) for arbitrary (X, U) and $(\tilde{X}, \tilde{U})$ now follow by the telescope summation. □

5. Further Sturmian separation theorems via comparative index

In this section we derive further Sturmian separation theorems for proper focal points of two conjoined bases of (H), as well as new and optimal bounds for the numbers of proper focal points of one conjoined basis of (H). These results are essentially based on Theorem 4.1. First we derive a formula, which relates the number of left proper focal points in $(a, b]$ and the number of right proper focal points in $[a, b)$ for one conjoined basis of (H).

Theorem 5.1. *Assume that (1.1) holds. Then for any conjoined basis (X, U) of (H) its numbers of left proper focal points in $(a, b]$ and right proper focal points in $[a, b)$ satisfy*

$$m_L(a, b] + \operatorname{rank} X(b) = m_R[a, b) + \operatorname{rank} X(a). \tag{5.1}$$

Proof. Let (X, U) be a conjoined basis of (H). By Proposition 1.1 we know that the kernel of (X, U) is piecewise constant on $[a, b]$, in particular, the numbers $m_L(a, b]$ and $m_R[a, b)$ are finite. Since by (1.5) and (1.6) we have

$$m_L(a, b] = m_L(a, b) + \operatorname{def} X(b) - \operatorname{def} X(b^-), \quad m_R[a, b) = m_R(a, b) + \operatorname{def} X(a) - \operatorname{def} X(a^+),$$

it follows that formula (5.1) is equivalent with

$$m_L(a, b) + \operatorname{rank} X(b^-) = m_R(a, b) + \operatorname{rank} X(a^+). \tag{5.2}$$

Therefore, the statement in the theorem will be proven once we show that (5.2) holds. Let $t_1, \dots, t_m \in (a, b)$ be the points where the kernel of $X(t)$ changes inside (a, b) and put $t_0 := a$ and $t_{m+1} := b$. Then $\operatorname{rank} X(t)$ is constant on each interval (t_j, t_{j+1}), i.e., we may define the quantity $r_j := \operatorname{rank} X(t)$ for $t \in (t_j, t_{j+1})$ and $j \in \{0, 1, \dots, m\}$. It follows that

$$m_L(t_{j+1}) = r_j - \operatorname{rank} X(t_{j+1}), \quad m_R(t_j) = r_j - \operatorname{rank} X(t_j), \quad j \in \{0, 1, \dots, m\}. \tag{5.3}$$

Consequently, we have by the telescope summation

$$\begin{aligned} m_L(a, b) - m_R(a, b) &= \sum_{j=1}^{m} [\, m_L(t_j) - m_R(t_j)\,] \\ &\overset{(5.3)}{=} \sum_{j=1}^{m} [\, r_{j-1} - \operatorname{rank} X(t_j) - r_j + \operatorname{rank} X(t_j)\,] \\ &= \sum_{j=1}^{m} (r_{j-1} - r_j) = r_0 - r_m = \operatorname{rank} X(a^+) - \operatorname{rank} X(b^-). \end{aligned}$$

Therefore, formula (5.2) and hence also (5.1) holds, which completes the proof. □

The statements in Proposition 1.4 can be reformulated as

$$\big|\, m_L(a,b] - \widetilde{m}_L(a,b] \,\big| \le n, \quad \big|\, m_R[a,b) - \widetilde{m}_R[a,b) \,\big| \le n. \tag{5.4}$$

In the following results we use the comparative index as a tool, which allows to improve the estimates in (5.4).

Theorem 5.2 *(Sturmian separation theorem). Assume that (1.1) holds. Then for any conjoined bases (X,U) and $(\tilde{X},\tilde{U})$ of (H) we have the estimates*

$$\big|\, m_L(a,b] - \widetilde{m}_L(a,b] \,\big| \le \min\big\{ \operatorname{rank} W,\ r(a,b),\ \tilde{r}(a,b) \big\} \le n, \tag{5.5}$$

$$\big|\, m_R[a,b) - \widetilde{m}_R[a,b) \,\big| \le \min\big\{ \operatorname{rank} W,\ r(a,b),\ \tilde{r}(a,b) \big\} \le n, \tag{5.6}$$

where W is the (constant) Wronskian of (X,U) and $(\tilde{X},\tilde{U})$ and

$$r(a,b) := \max\{\operatorname{rank} X(a), \operatorname{rank} X(b)\}, \quad \tilde{r}(a,b) := \max\{\operatorname{rank} \tilde{X}(a), \operatorname{rank} \tilde{X}(b)\}. \tag{5.7}$$

Proof. By formula (4.6) in Theorem 4.1 we know that the left-hand side of (5.5) is equal to $|\mu(b)-\mu(a)|$, where $\mu(t)$ is the abbreviation for $\mu\big(Y(t), \tilde{Y}(t)\big)$. From (2.8) we know that $\mu(a) \le \min\{\operatorname{rank} W, \operatorname{rank} \tilde{X}(a)\}$ and $\mu(b) \le \min\{\operatorname{rank} W, \operatorname{rank} \tilde{X}(b)\}$, which yields that

$$|\mu(b) - \mu(a)| \le \min\{\operatorname{rank} W,\ \tilde{r}(a,b)\}, \tag{5.8}$$

where $\tilde{r}(a,b)$ is given in (5.7). If we now switch the roles of the conjoined bases (X,U) and $(\tilde{X},\tilde{U})$, abbreviate $\mu\big(\tilde{Y}(t), Y(t)\big)$ as $\tilde{\mu}(t)$, and use that the Wronskian of $(\tilde{X},\tilde{U})$ and (X,U) is equal to $-W^T$, then the formula $\mu(t) + \tilde{\mu}(t) = \operatorname{rank} W$ in (2.7) yields

$$|\mu(b) - \mu(a)| = |\tilde{\mu}(b) - \tilde{\mu}(a)| \overset{(5.8)}{\le} \min\{\operatorname{rank} W,\ r(a,b)\} \tag{5.9}$$

with $r(a,b)$ given again in (5.7). By combining (5.8) and (5.9) we obtain the estimate in (5.5). Formula (5.6) is proven in a similar way by using (4.7) and the properties of the dual comparative index in (2.7) and (2.8). $\square$

Similarly with (4.2)–(4.5) we introduce the following notation involving the principal solutions $(\hat{X}_a, \hat{U}_a)$ and $(\hat{X}_b, \hat{U}_b)$ of (H) at the points a and b, respectively:

$$\widehat{m}_{La}(a,b] := \text{ the number of left proper focal points of } (\hat{X}_a, \hat{U}_a) \text{ in } (a,b], \tag{5.10}$$

$$\widehat{m}_{Ra}[a,b) := \text{ the number of right proper focal points of } (\hat{X}_a, \hat{U}_a) \text{ in } [a,b), \tag{5.11}$$

$$\widehat{m}_{Lb}(a,b] := \text{ the number of left proper focal points of } (\hat{X}_b, \hat{U}_b) \text{ in } (a,b], \tag{5.12}$$

$$\widehat{m}_{Rb}[a,b) := \text{ the number of right proper focal points of } (\hat{X}_b, \hat{U}_b) \text{ in } [a,b). \tag{5.13}$$

These numbers turn out to be essential parameters in the Sturmian theory of system (H) on the interval $[a,b]$.

The next two results relate the numbers of left or right proper focal points of the principal solutions $(\hat{X}_a, \hat{U}_a)$ and $(\hat{X}_b, \hat{U}_b)$ in $(a, b]$ and $[a, b)$. More precisely, in (5.14) we show how to compute the number of left proper focal points of $(\hat{X}_b, \hat{U}_b)$ in $(a, b]$ in terms of the left proper focal points of $(\hat{X}_a, \hat{U}_a)$ in $(a, b]$, while in (5.15) we show the same for the right proper focal points of $(\hat{X}_a, \hat{U}_a)$ in $[a, b)$ in terms of the right proper focal points of $(\hat{X}_b, \hat{U}_b)$ in $[a, b)$.

Theorem 5.3. *Assume that (1.1) holds. With the notation in (5.10)–(5.13) we have for the left and right focal points of the principal solutions $(\hat{X}_a, \hat{U}_a)$ and $(\hat{X}_b, \hat{U}_b)$ in $(a, b]$ and $[a, b)$, respectively, the equalities*

$$\widehat{m}_{Lb}(a, b] = \widehat{m}_{La}(a, b] + \operatorname{rank} \hat{X}_a(b) = \widehat{m}_{La}(a, b] + \operatorname{rank} \hat{X}_b(a), \tag{5.14}$$

$$\widehat{m}_{Ra}[a, b) = \widehat{m}_{Rb}[a, b) + \operatorname{rank} \hat{X}_b(a) = \widehat{m}_{Rb}[a, b) + \operatorname{rank} \hat{X}_a(b). \tag{5.15}$$

Proof. We first apply Theorem 4.1 with $(X, U) := (\hat{X}_b, \hat{U}_b)$ and $(\tilde{X}, \tilde{U}) := (\hat{X}_a, \hat{U}_a)$ to get

$$\widehat{m}_{Lb}(a, b] - \widehat{m}_{La}(a, b] \overset{(4.6)}{=} \mu\big(E, \hat{Y}_a(b)\big) - \mu\big(\hat{Y}_b(a), E\big) \overset{(2.9)}{=} \operatorname{rank} \hat{X}_a(b),$$

and then again Theorem 4.1 with $(X, U) := (\hat{X}_a, \hat{U}_a)$ and $(\tilde{X}, \tilde{U}) := (\hat{X}_b, \hat{U}_b)$ to get

$$\widehat{m}_{Ra}[a, b) - \widehat{m}_{Rb}[a, b) \overset{(4.7)}{=} \mu^*\big(E, \hat{Y}_b(a)\big) - \mu^*\big(\hat{Y}_a(b), E\big) \overset{(2.9)}{=} \operatorname{rank} \hat{X}_b(a).$$

This shows the first equalities in (5.14) and (5.15). For the second equalities we note that $\operatorname{rank} \hat{X}_a(b) = \operatorname{rank} \hat{X}_b(a)$, which follows from the relation $\hat{X}_b(a) = -\hat{X}_a^T(b)$ obtained by the evaluation of the constant Wronskian of $(\hat{X}_a, \hat{U}_a)$ and $(\hat{X}_b, \hat{U}_b)$ at $t = a$ and $t = b$. □

Corollary 5.4. *Assume that (1.1) holds. With the notation in (5.10)–(5.13) we have*

$$\widehat{m}_{Ra}[a, b) = \widehat{m}_{Lb}(a, b], \quad \widehat{m}_{La}(a, b] = \widehat{m}_{Rb}[a, b). \tag{5.16}$$

Proof. By using Theorem 5.1 with $(X, U) := (\hat{X}_a, \hat{U}_a)$ we obtain that

$$\widehat{m}_{La}(a, b] + \operatorname{rank} \hat{X}_a(b) \overset{(5.1)}{=} \widehat{m}_{Ra}[a, b). \tag{5.17}$$

Upon inserting (5.17) into (5.14) we obtain the first equality in (5.16). Similarly, we have

$$\widehat{m}_{La}(a, b] \overset{(5.17)}{=} \widehat{m}_{Ra}[a, b) - \operatorname{rank} \hat{X}_a(b) \overset{(5.15)}{=} \widehat{m}_{Rb}[a, b),$$

which proves the second equality in (5.16). □

Remark 5.5. The equation $\widehat{m}_{La}(a, b] = \widehat{m}_{Rb}[a, b)$ in (5.16) is a reformulation of the statement in Proposition 1.2, hence it is already known, see also [24, Corollary 4.8]. The proof in the latter reference is based on the oscillation theorem for the associated eigenvalue problem and its time-reversed form. In this paper we present a different proof, which depends on the properties of the comparative index rather than on the properties of the eigenvalues. Therefore, the present study is self-contained and independent on the results in [24].

In the following result we derive new and optimal bounds for the numbers of left and right proper focal points of any conjoined basis (X, U) in $(a, b]$ and $[a, b)$, respectively. These bounds are optimal in a sense that they are formulated in terms of quantities, which do not depend on the chosen conjoined basis (X, U). More precisely, they are formulated in terms of the numbers of left and right proper focal points of principal solutions $(\hat{X}_a, \hat{U}_a)$ and $(\hat{X}_b, \hat{U}_b)$.

Theorem 5.6 *(Sturmian separation theorem). Assume that (1.1) holds. Then for any conjoined basis (X, U) of (H) we have the inequalities*

$$\widehat{m}_{La}(a, b] \le m_L(a, b] \le \widehat{m}_{Lb}(a, b] = \widehat{m}_{La}(a, b] + \operatorname{rank} \hat{X}_a(b), \tag{5.18}$$

$$\widehat{m}_{Rb}[a, b) \le m_R[a, b) \le \widehat{m}_{Ra}[a, b) = \widehat{m}_{Rb}[a, b) + \operatorname{rank} \hat{X}_b(a). \tag{5.19}$$

Proof. We apply Theorem 4.1 with $(\tilde{X}, \tilde{U}) := (\hat{X}_a, \hat{U}_a)$ or $(\tilde{X}, \tilde{U}) := (\hat{X}_b, \hat{U}_b)$ and use the fact that $\hat{Y}_a(a) = E = \hat{Y}_b(b)$ and $\mu(Y, E) = 0 = \mu^*(Y, E)$ by (2.9). Since the comparative index is nonnegative, it follows that

$$m_L(a, b] - \widehat{m}_{La}(a, b] \overset{(4.6)}{=} \mu\big(Y(b), \hat{Y}_a(b)\big) - \mu\big(Y(a), E\big) \overset{(2.9)}{=} \mu\big(Y(b), \hat{Y}_a(b)\big) \ge 0, \tag{5.20}$$

$$m_L(a, b] - \widehat{m}_{Lb}(a, b] \overset{(4.6)}{=} \mu\big(Y(b), E\big) - \mu\big(Y(a), \hat{Y}_b(a)\big) \overset{(2.9)}{=} -\mu\big(Y(a), \hat{Y}_b(a)\big) \le 0. \tag{5.21}$$

Similarly, for the right proper focal points we have

$$m_R[a, b) - \widehat{m}_{Rb}[a, b) \overset{(4.7)}{=} \mu^*\big(Y(a), \hat{Y}_b(a)\big) - \mu^*\big(Y(b), E\big) \overset{(2.9)}{=} \mu^*\big(Y(a), \hat{Y}_b(a)\big) \ge 0, \tag{5.22}$$

$$m_R[a, b) - \widehat{m}_{Ra}[a, b) \overset{(4.7)}{=} \mu^*\big(Y(a), E\big) - \mu^*\big(Y(b), \hat{Y}_a(b)\big) \overset{(2.9)}{=} -\mu^*\big(Y(b), \hat{Y}_a(b)\big) \le 0. \tag{5.23}$$

These calculations show that the first two inequalities in (5.18) and (5.19) are satisfied. The last equalities in (5.18) and (5.19) follow directly from Theorem 5.3. □

Remark 5.7. According to (5.16) in Corollary 5.4, the lower bounds in (5.18) and (5.19) are the same, as well as the upper bounds in (5.18) and (5.19) are the same. Moreover, these lower and upper bounds are independent on the conjoined basis (X, U). Since these bounds are attained for the specific choices of $(X, U) := (\hat{X}_a, \hat{U}_a)$ and $(X, U) := (\hat{X}_b, \hat{U}_b)$, the inequalities in (5.18) and (5.19) cannot be improved – in the sense that the estimates (5.18) and (5.19) are satisfied *for all* conjoined bases (X, U) of (H).

The results in Theorem 5.6 yield another improvement of the estimates in (5.4). In particular, in contrast with Theorem 5.2 we obtain the estimates, which do not depend on the chosen conjoined bases (X, U) and $(\tilde{X}, \tilde{U})$. In addition, it allows to compare the numbers of left proper focal points of (X, U) and right proper focal points of $(\tilde{X}, \tilde{U})$ and vice versa.

Corollary 5.8 *(Sturmian separation theorem). Assume that (1.1) holds. Then for any conjoined bases (X, U) and $(\tilde{X}, \tilde{U})$ of (H) we have the estimates*

$$\big| m_L(a,b] - \widetilde{m}_L(a,b] \big| \le \operatorname{rank} \hat{X}_a(b) = \operatorname{rank} \hat{X}_b(a) \le n, \tag{5.24}$$

$$\big| m_R[a,b) - \widetilde{m}_R[a,b) \big| \le \operatorname{rank} \hat{X}_b(a) = \operatorname{rank} \hat{X}_a(b) \le n, \tag{5.25}$$

$$\big| m_L(a,b] - \widetilde{m}_R[a,b) \big| \le \operatorname{rank} \hat{X}_a(b) = \operatorname{rank} \hat{X}_b(a) \le n. \tag{5.26}$$

In particular, for one conjoined basis (X, U) of (H) we have

$$\big| m_L(a,b] - m_R[a,b) \big| = \big| \operatorname{rank} X(a) - \operatorname{rank} X(b) \big| \le \operatorname{rank} \hat{X}_a(b) = \operatorname{rank} \hat{X}_b(a) \le n. \tag{5.27}$$

Proof. Inequality (5.24) follows from (5.18) in Theorem 5.6 applied to (X, U) and $(\tilde{X}, \tilde{U})$. Similarly, (5.25) follows from (5.19) applied to (X, U) and $(\tilde{X}, \tilde{U})$. Next we apply inequality (5.18) to (X, U) and inequality (5.19) to $(\tilde{X}, \tilde{U})$ and use Corollary 5.4 to obtain (5.26). Finally, estimate (5.27) follows from Theorem 5.1 and from inequality (5.26) with $(\tilde{X}, \tilde{U}) := (X, U)$. □

In the proof of Theorem 5.6 we derive the exact formulas

$$m_L(a,b] = \widehat{m}_{La}(a,b] + \mu\big(Y(b), \hat{Y}_a(b)\big), \quad m_R[a,b) = \widehat{m}_{Rb}[a,b) + \mu^*\big(Y(a), \hat{Y}_b(a)\big), \tag{5.28}$$

which show how to calculate the number of left or right proper focal points of an arbitrary conjoined basis (X, U) of (H) as a sum of a quantity which does not depend on (X, U) and the comparative index of (X, U) with $(\hat{X}_a, \hat{U}_a)$ at $t = b$, or the dual comparative index of (X, U) with $(\hat{X}_b, \hat{U}_b)$ at $t = a$. The formulas in (5.28) then highlight the importance of the comparative index in the Sturmian theory of linear Hamiltonian systems (H).

Formulas (5.28) are especially important for theoretical investigations about the proper focal points of (X, U). For practical purposes, e.g. in the oscillation theory, it is more convenient to have estimates for the numbers $m_L(a,b]$ and $m_R[a,b)$, which do not explicitly involve the possible complicated evaluation of the comparative index. In Theorem 5.9 below we present such estimates of $m_L(a,b]$ and $m_R[a,b)$. At the same time we show that the universal lower and upper bounds for $m_L(a,b]$ and $m_R[a,b)$ in Theorem 5.6 can be improved for some particular choice of (X, U).

Theorem 5.9 *(Sturmian separation theorem). Assume that (1.1) holds. Then for any conjoined basis (X, U) of (H) we have the inequalities*

$$\widehat{m}_{La}(a,b] + \operatorname{rank} \hat{X}_a(b) - \min\{\operatorname{rank} \hat{X}_a(b),\ \operatorname{rank} X(b)\} \le m_L(a,b], \tag{5.29}$$

$$\widehat{m}_{La}(a,b] + \operatorname{rank} X(a) - \min\{\operatorname{rank} X(a),\ \operatorname{rank} X(b)\} \le m_L(a,b], \tag{5.30}$$

$$m_L(a,b] \le \widehat{m}_{La}(a,b] + \min\{\operatorname{rank} X(a),\ \operatorname{rank} \hat{X}_a(b)\}, \tag{5.31}$$

and

$$\widehat{m}_{Rb}[a,b) + \operatorname{rank} \hat{X}_b(a) - \min\{\operatorname{rank} \hat{X}_b(a),\ \operatorname{rank} X(a)\} \le m_R[a,b), \tag{5.32}$$

$$\widehat{m}_{Rb}[a,b) + \operatorname{rank} X(b) - \min\{\operatorname{rank} X(b),\ \operatorname{rank} X(a)\} \le m_R[a,b), \tag{5.33}$$

$$m_R[a,b) \le \widehat{m}_{Rb}[a,b) + \min\{\operatorname{rank} X(b),\ \operatorname{rank} \hat{X}_b(a)\}. \tag{5.34}$$

Proof. The estimates in (5.29) and (5.30) follow from a combination of (5.20) and Lemma 2.3, in which we take $Y := Y(b)$, $\tilde{Y} := \hat{Y}_a(b)$, and $W := X^T(a)$ being the Wronskian of (X,U) and $(\hat{X}_a, \hat{U}_a)$. With the same notation, the result in (5.31) follows from (5.20) and from the first condition in (2.8). In a similar way we obtain (5.32)–(5.34) from (5.22) and from Lemma 2.3 and the second condition in (2.8), in which we take $Y := Y(a)$, $\tilde{Y} := \hat{Y}_b(a)$, and $W := X^T(b)$ being the Wronskian of (X,U) and $(\hat{X}_b, \hat{U}_b)$. □

The estimates in Theorems 5.6 and 5.9 yield the following improvement of Corollary 5.8.

Corollary 5.10 *(Sturmian separation theorem). Assume that (1.1) holds. Then for any conjoined bases (X,U) and $(\tilde{X},\tilde{U})$ of (H) we have the estimates*

$$\left| m_L(a,b] - \widetilde{m}_L(a,b] \right| \le \min\left\{ \max\{\operatorname{rank} X(a), \operatorname{rank} \tilde{X}(a)\},\ \operatorname{rank} \hat{X}_a(b) \right\} \le n, \tag{5.35}$$

$$\left| m_R[a,b) - \widetilde{m}_R[a,b) \right| \le \min\left\{ \max\{\operatorname{rank} X(b), \operatorname{rank} \tilde{X}(b)\},\ \operatorname{rank} \hat{X}_b(a) \right\} \le n, \tag{5.36}$$

$$\left| m_L(a,b] - \widetilde{m}_R[a,b) \right| \le \min\left\{ \max\{\operatorname{rank} X(a), \operatorname{rank} \tilde{X}(b)\},\ \operatorname{rank} \hat{X}_a(b) \right\} \le n. \tag{5.37}$$

In particular, for one conjoined basis (X,U) of (H) we have

$$\left| m_L(a,b] - m_R[a,b) \right| \le \min\left\{ \max\{\operatorname{rank} X(a), \operatorname{rank} X(b)\},\ \operatorname{rank} \hat{X}_a(b) \right\} \le n. \tag{5.38}$$

Proof. From the lower bound in (5.18) and from the upper bound in (5.31) applied to (X,U) and $(\tilde{X},\tilde{U})$ we get the estimate

$$\left| m_L(a,b] - \widetilde{m}_L(a,b] \right| \le \max\left\{ \min\{\operatorname{rank} X(a),\ \operatorname{rank} \hat{X}_a(b)\},\ \min\{\operatorname{rank} \tilde{X}(a),\ \operatorname{rank} \hat{X}_a(b)\} \right\}.$$

Inequality (5.35) now follows from the fact that for any real numbers x, y, z we have the equality $\max\{\min\{x,z\},\ \min\{y,z\}\} = \min\{\max\{x,y\},\ z\}$. In a similar way we obtain (5.36) from the lower bound in (5.19) and from the upper bound in (5.34) applied to (X,U) and $(\tilde{X},\tilde{U})$. Also, (5.37) follows by a combination of (5.18) and (5.31) applied to (X,U) and of (5.19) and (5.34) applied to $(\tilde{X},\tilde{U})$. Finally, (5.38) is a consequence of (5.37) for the choice $(\tilde{X},\tilde{U}) := (X,U)$. Note also that (5.38) can be obtained from (5.27) in Corollary 5.8, if we realize that $|\operatorname{rank} X(a) - \operatorname{rank} X(b)| \le \max\{\operatorname{rank} X(a), \operatorname{rank} X(b)\}$. □

Remark 5.11. The results in Corollary 5.10 have interesting consequences. Namely, it is possible to obtain an estimate for the difference of the left proper focal points of two conjoined bases (X,U) and $(\tilde{X},\tilde{U})$ in the interval $(a,b]$, which does not depend on the values at the right endpoint b. More precisely, inequality (5.35) implies that

$$\left| m_L(a,b] - \widetilde{m}_L(a,b] \right| \le \max\{\operatorname{rank} X(a), \operatorname{rank} \tilde{X}(a)\} \le n. \tag{5.39}$$

Similarly, inequality (5.36) yields the estimate

$$\left| m_R[a,b) - \widetilde{m}_R[a,b) \right| \leq \max\{\operatorname{rank} X(b), \operatorname{rank} \tilde{X}(b)\} \leq n, \tag{5.40}$$

which does not depend on the values at the left endpoint a. The results in (5.39) and (5.40) allow to compare the numbers of proper focal points of (X,U) and $(\tilde{X},\tilde{U})$ in unbounded intervals, when the system (H) is nonoscillatory, see also Remark 7.6.

6. Continuity and limit properties of comparative index

In this section we derive two results regarding the continuity and limit properties of the comparative index and a connection of the comparative index (or its one-sided limits) with the multiplicities of proper focal points of a conjoined basis of (H) at a point t_0.

Theorem 6.1. *Assume that (1.1) holds. Then for any two conjoined bases (X,U) and $(\hat{X},\hat{U})$ of (H) the following properties are satisfied.*

(i) *The comparative index $\mu\big(Y(t),\tilde{Y}(t)\big)$ is piecewise constant on $[a,b]$ and right continuous on $[a,b)$. In addition,*

$$\lim_{t\to t_0^-} \mu\big(Y(t),\tilde{Y}(t)\big) = \mu\big(Y(t_0),\tilde{Y}(t_0)\big) - m_L(t_0) + \widetilde{m}_L(t_0), \quad t_0 \in (a,b], \tag{6.1}$$

and $\mu\big(Y(t),\tilde{Y}(t)\big)$ is not left continuous at $t_0 \in (a,b]$ if and only if $m_L(t_0) \neq \widetilde{m}_L(t_0)$.

(ii) *The dual comparative index $\mu^*\big(Y(t),\tilde{Y}(t)\big)$ is piecewise constant on $[a,b]$ and left continuous on $(a,b]$. In addition,*

$$\lim_{t\to t_0^+} \mu^*\big(Y(t),\tilde{Y}(t)\big) = \mu^*\big(Y(t_0),\tilde{Y}(t_0)\big) - m_R(t_0) + \widetilde{m}_R(t_0), \quad t_0 \in [a,b), \tag{6.2}$$

and $\mu^*\big(Y(t),\tilde{Y}(t)\big)$ is not right continuous at $t_0 \in [a,b)$ if and only if $m_R(t_0) \neq \widetilde{m}_R(t_0)$.

Proof. Let $t_0 \in (a,b]$ be fixed and let $t \in (a,t_0)$ be close to t_0. We apply formula (4.6) in Theorem 4.1 with $a := t$ and $b := t_0$ and let $t \nearrow t_0$. We have

$$\begin{aligned} m_L(t_0) - \widetilde{m}_L(t_0) &= \lim_{t\to t_0^-} m_L(t,t_0] - \lim_{t\to t_0^-} \widetilde{m}_L(t,t_0] = \lim_{t\to t_0^-} \big\{ m_L(t,t_0] - \widetilde{m}_L(t,t_0] \big\} \\ &\overset{(4.6)}{=} \mu\big(Y(t_0),\tilde{Y}(t_0)\big) - \lim_{t\to t_0^-} \mu\big(Y(t),\tilde{Y}(t)\big). \end{aligned} \tag{6.3}$$

Similarly, if $t_0 \in [a,b)$ is fixed and $t \in (t_0,b)$ is close to t_0, then we may assume without loss of generality that $\operatorname{Ker} X(s)$ and $\operatorname{Ker} \tilde{X}(s)$ are constant for $s \in (t_0,t)$, so that both $m_L(t_0,t] = 0$ and $\widetilde{m}_L(t_0,t] = 0$. Hence, by formula (4.6) with $a := t_0$ and $b := t$ and taking $t \searrow t_0$ we have

$$0 = \lim_{t\to t_0^+} \big\{ m_L(t_0,t] - \widetilde{m}_L(t_0,t] \big\} \overset{(4.6)}{=} \lim_{t\to t_0^+} \mu\big(Y(t),\tilde{Y}(t)\big) - \mu\big(Y(t_0),\tilde{Y}(t_0)\big). \tag{6.4}$$

Formulas (6.3) and (6.4) show that the one-sided limits of $\mu\big(Y(t),\tilde{Y}(t)\big)$ exist and that

$$\lim_{t\to t_0^+} \mu\big(Y(t), \tilde{Y}(t)\big) = \mu\big(Y(t_0), \tilde{Y}(t_0)\big), \quad t_0 \in [a, b), \tag{6.5}$$

as well as (6.1) hold. Therefore, the comparative index $\mu\big(Y(t), \tilde{Y}(t)\big)$ is right continuous on $[a, b)$ and it is not left continuous exactly at points $t_0 \in (a, b]$, where $m_L(t_0) \neq \widetilde{m}_L(t_0)$. Moreover, since $\mu\big(Y(t), \tilde{Y}(t)\big)$ attains only integer values, it follows from (6.1) and (6.5) that $\mu\big(Y(t), \tilde{Y}(t)\big)$ is constant on some left neighborhood of every $t_0 \in (a, b]$ and on some right neighborhood of every $t_0 \in [a, b)$. These facts together with the compactness of $[a, b]$ imply that $\mu\big(Y(t), \tilde{Y}(t)\big)$ is piecewise constant on $[a, b]$, which completes the proof of part (i). For part (ii) we proceed in a similar way by applying formula (4.7) in Theorem 4.1, once for $t_0 \in (a, b]$ with $a := t$, $b := t_0$, and $t \nearrow t_0$ to get $m_R[t, t_0) = 0 = \widetilde{m}_R[t, t_0)$ and

$$0 = \lim_{t\to t_0^-} \big\{m_R[t, t_0) - \widetilde{m}_R[t, t_0)\big\} \overset{(4.7)}{=} \lim_{t\to t_0^-} \mu^*\big(Y(t), \tilde{Y}(t)\big) - \mu^*\big(Y(t_0), \tilde{Y}(t_0)\big), \tag{6.6}$$

and once for $t_0 \in [a, b)$ with $a := t_0$, $b := t$, and $t \searrow t_0$ to get

$$\begin{aligned} m_R(t_0) - \widetilde{m}_R(t_0) &= \lim_{t\to t_0^+} m_R[t_0, t) - \lim_{t\to t_0^+} \widetilde{m}_R[t_0, t) = \lim_{t\to t_0^+} \big\{m_R[t_0, t) - \widetilde{m}_R[t_0, t)\big\} \\ &\overset{(4.7)}{=} \mu^*\big(Y(t_0), \tilde{Y}(t_0)\big) - \lim_{t\to t_0^+} \mu^*\big(Y(t), \tilde{Y}(t)\big). \end{aligned} \tag{6.7}$$

Equations (6.6) and (6.7) then show the statements in part (ii). □

Remark 6.2. Since $\mu\big(Y(t), \tilde{Y}(t)\big)$ and $\mu^*\big(Y(t), \tilde{Y}(t)\big)$ attain only nonnegative integer values, the existence of their one-sided limits showed in the proof of Theorem 6.1 is equivalent with the conditions

$$\begin{aligned} \mu\big(Y(t), \tilde{Y}(t)\big) &\equiv \mu\big(Y(t_0), \tilde{Y}(t_0)\big) - m_L(t_0) + \widetilde{m}_L(t_0) \quad \text{for all } t \in (t_0 - \varepsilon, t_0), \\ \mu\big(Y(t), \tilde{Y}(t)\big) &\equiv \mu\big(Y(t_0), \tilde{Y}(t_0)\big) \quad \text{for all } t \in (t_0, t_0 + \varepsilon), \\ \mu^*\big(Y(t), \tilde{Y}(t)\big) &\equiv \mu^*\big(Y(t_0), \tilde{Y}(t_0)\big) \quad \text{for all } t \in (t_0 - \varepsilon, t_0), \\ \mu^*\big(Y(t), \tilde{Y}(t)\big) &\equiv \mu^*\big(Y(t_0), \tilde{Y}(t_0)\big) - m_R(t_0) + \widetilde{m}_R(t_0) \quad \text{for all } t \in (t_0, t_0 + \varepsilon), \end{aligned}$$

in all four cases for some $\varepsilon > 0$ depending on the chosen point t_0.

The next result shows how to compute the multiplicities in (1.5) and (1.6) of left and right proper focal points of (X, U) at some point t_0 by a limit involving the comparative index.

Theorem 6.3. *Assume that (1.1) holds. Let (X, U) be a conjoined basis of (H) and let $(\hat{X}_t, \hat{U}_t)$ be the principal solution of (H) at the point t, i.e., (3.1) holds with $s = t$. Then with the notation in (4.1) we have*

$$m_L(t_0) = \lim_{t\to t_0^-} \mu\big(Y(t_0), \hat{Y}_t(t_0)\big), \quad t_0 \in (a, b], \tag{6.8}$$

$$m_R(t_0) = \lim_{t\to t_0^+} \mu^*\big(Y(t_0), \hat{Y}_t(t_0)\big), \quad t_0 \in [a, b). \tag{6.9}$$

Proof. Let $t_0 \in (a, b]$ be fixed. For (6.8) we apply formula (4.6) on the interval $[t, t_0]$ with $(\tilde{X}, \tilde{U}) := (\hat{X}_t, \hat{U}_t)$ being the principal solution at t, together with the notation (5.10) and (5.13) on $[t, t_0]$. First we observe that the principal solution $(\hat{X}_{t_0}, \hat{U}_{t_0})$ of (H) at t_0 has no right proper focal points in $[t, t_0)$ when $t < t_0$ is close enough to t_0, since the kernel of $\hat{X}_{t_0}(s)$ is constant for $s \in [t, t_0)$ in this case. Therefore, $\widehat{m}_{Rt_0}[t, t_0) = 0$ for all $t < t_0$ close enough to t_0. This implies by (5.16) that also

$$\widehat{m}_{Lt}(t, t_0] = \widehat{m}_{Rt_0}[t, t_0) = 0 \quad \text{for all } t < t_0 \text{ close enough to } t_0. \tag{6.10}$$

By using $\hat{Y}_t(t) = E$ and formula (2.9) we obtain that $\mu\big(Y(t), E\big) = 0$ and hence,

$$\begin{aligned} m_L(t_0) &= \lim_{t \to t_0^-} m_L(t, t_0] \overset{(6.10)}{=} \lim_{t \to t_0^-} \big\{ m_L(t, t_0] - \widehat{m}_{Lt}(t, t_0] \big\} \\ &\overset{(4.6)}{=} \lim_{t \to t_0^-} \big\{ \mu\big(Y(t_0), \hat{Y}_t(t_0)\big) - \mu\big(Y(t), E\big) \big\} = \lim_{t \to t_0^-} \mu\big(Y(t_0), \hat{Y}_t(t_0)\big), \end{aligned}$$

which proves (6.8). In a similar way, if $t_0 \in [a, b)$ is fixed, then the principal solution $(\hat{X}_{t_0}, \hat{U}_{t_0})$ of (H) has no left proper focal points in $(t_0, t]$ when $t > t_0$ is close to t_0, since the kernel of $\hat{X}_{t_0}(s)$ is constant for $s \in (t_0, t]$ in this case. Therefore, by (5.16) we obtain

$$\widehat{m}_{Rt}[t_0, t) = \widehat{m}_{Lt_0}(t_0, t] = 0 \quad \text{for all } t > t_0 \text{ close enough to } t_0. \tag{6.11}$$

By using formula (2.9) we then obtain that $\mu^*\big(Y(t), E\big) = 0$ and hence,

$$\begin{aligned} m_R(t_0) &= \lim_{t \to t_0^+} m_R[t_0, t) \overset{(6.11)}{=} \lim_{t \to t_0^+} \big\{ m_R[t_0, t) - \widehat{m}_{Rt}[t_0, t) \big\} \\ &\overset{(4.7)}{=} \lim_{t \to t_0^+} \big\{ \mu^*\big(Y(t_0), \hat{Y}_t(t_0)\big) - \mu^*\big(Y(t), E\big) \big\} = \lim_{t \to t_0^+} \mu^*\big(Y(t_0), \hat{Y}_t(t_0)\big). \end{aligned}$$

This shows that (6.9) holds and the proof is complete. □

Remark 6.4. Formulas (6.10) and (6.11) in the above proof have the following interpretation. There exists $\varepsilon > 0$ such that for all $t \in [t_0 - \varepsilon, t_0)$ the principal solution $(\hat{X}_t, \hat{U}_t)$ of (H) at the point t has nonincreasing kernel on the interval $(t, t_0]$, and for all $t \in (t_0, t_0 + \varepsilon]$ the principal solution $(\hat{X}_t, \hat{U}_t)$ of (H) at the point t has nondecreasing kernel on the interval $[t_0, t)$. Therefore, by (5.15) in Theorem 5.3 together with $\widehat{m}_{Rt_0}[t, t_0) = 0$, $\hat{X}_t(t_0) = -\hat{X}_{t_0}^T(t)$, and (3.20) we obtain

$$\widehat{m}_{Rt}[t, t_0) \overset{(5.15)}{=} \operatorname{rank} \hat{X}_t(t_0) = \operatorname{rank} \hat{X}_{t_0}(t) \overset{(3.20)}{=} n - d[t, t_0], \quad t \in [t_0 - \varepsilon, t_0). \tag{6.12}$$

Similarly, from (5.14) and (3.19) together with $\widehat{m}_{Lt_0}(t_0, t] = 0$ we obtain

$$\widehat{m}_{Lt}(t_0, t] \overset{(5.14)}{=} \operatorname{rank} \hat{X}_t(t_0) = \operatorname{rank} \hat{X}_{t_0}(t) \overset{(3.19)}{=} n - d[t_0, t], \quad t \in (t_0, t_0 + \varepsilon]. \tag{6.13}$$

In Theorem 6.3 we established the existence of the left-hand limit at t_0 of the comparative index $\mu\big(Y(t_0), \hat{Y}_t(t_0)\big)$ and the right-hand limit at t_0 of the dual comparative index $\mu^*\big(Y(t_0), \hat{Y}_t(t_0)\big)$. In the next result we show that these comparative indices have limits at the point t_0 also from the opposite sides. It is surprising that the values of these limits then depend on the abnormality of system (H) in the corresponding left or right neighborhood of the point t_0. We recall from Remark 3.4 the notation $d_{t_0}^{\pm}$ for the maximal order of abnormality of system (H) at the point t_0 from the right and left.

Theorem 6.5. *Assume that (1.1) holds. Let (X, U) be a conjoined basis of (H) and let $(\hat{X}_t, \hat{U}_t)$ be the principal solution of (H) at the point t. Then*

$$\lim_{t\to t_0^+} \mu\big(Y(t_0), \hat{Y}_t(t_0)\big) = n - d_{t_0}^+, \quad t_0 \in [a, b), \tag{6.14}$$

$$\lim_{t\to t_0^-} \mu^*\big(Y(t_0), \hat{Y}_t(t_0)\big) = n - d_{t_0}^-, \quad t_0 \in (a, b], \tag{6.15}$$

where the numbers $d_{t_0}^+$ and $d_{t_0}^-$ are defined in (3.17) and (3.18).

Proof. Let $t_0, t \in [a, b]$ be given. Since the Wronskian of (X, U) and $(\hat{X}_t, \hat{U}_t)$ is equal to $X^T(t)$, it follows from properties (2.6) and (2.7) that

$$\mu\big(Y(t_0), \hat{Y}_t(t_0)\big) + \mu^*\big(Y(t_0), \hat{Y}_t(t_0)\big) = \operatorname{rank} X(t) + \operatorname{rank} \hat{X}_t(t_0) - \operatorname{rank} X(t_0). \tag{6.16}$$

Assume that $t_0, t \in [a, b)$ with $t_0 < t$. By (6.13) in Remark 6.4 we know that $\operatorname{rank} \hat{X}_t(t_0) = n - d[t_0, t]$ when t is sufficiently close to t_0. By Proposition 1.1 we also know that $\operatorname{rank} X(s) \equiv \operatorname{rank} X(t_0^+)$ is constant on $(t_0, t]$ for t close to t_0. Combining these facts with the limit in (6.9) in Theorem 6.3 we obtain from (6.16) that the limit of $\mu\big(Y(t_0), \hat{Y}_t(t_0)\big)$ as $t \to t_0^+$ exists and

$$\begin{aligned}
\lim_{t\to t_0^+} \mu\big(Y(t_0), \hat{Y}_t(t_0)\big) &= \operatorname{rank} X(t_0^+) + \lim_{t\to t_0^+} \big(n - d[t_0, t]\big) - \operatorname{rank} X(t_0) \\
&\quad - \lim_{t\to t_0^+} \mu^*\big(Y(t_0), \hat{Y}_t(t_0)\big) \\
&\overset{(3.17)}{=} \operatorname{rank} X(t_0^+) - \operatorname{rank} X(t_0) + n - d_{t_0}^+ - m_R(t_0) \overset{(1.6)}{=} n - d_{t_0}^+.
\end{aligned}$$

Therefore, equality (6.14) holds. Next we assume that $t_0, t \in (a, b]$ with $t < t_0$. Then similarly as above we obtain from (6.16) by using (6.12), (6.8), (3.18), and (1.5) that the limit of $\mu^*\big(Y(t_0), \hat{Y}_t(t_0)\big)$ as $t \to t_0^-$ also exists and

$$\begin{aligned}
\lim_{t\to t_0^-} \mu^*\big(Y(t_0), \hat{Y}_t(t_0)\big) &= \operatorname{rank} X(t_0^-) + \lim_{t\to t_0^-} \big(n - d[t, t_0]\big) - \operatorname{rank} X(t_0) \\
&\quad - \lim_{t\to t_0^-} \mu\big(Y(t_0), \hat{Y}_t(t_0)\big) \\
&\overset{(3.18)}{=} \operatorname{rank} X(t_0^-) - \operatorname{rank} X(t_0) + n - d_{t_0}^- - m_L(t_0) \overset{(1.5)}{=} n - d_{t_0}^-.
\end{aligned}$$

Therefore, equality (6.15) holds and the proof is complete. $\square$

Remark 6.6. Since the values of the comparative index $\mu\big(Y(t_0), \hat{Y}_t(t_0)\big)$ and the dual comparative index $\mu^*\big(Y(t_0), \hat{Y}_t(t_0)\big)$ are nonnegative integers, the existence of the limits in (6.8), (6.14) and in (6.9), (6.15) is equivalent, respectively, to the statements

$$\mu\big(Y(t_0), \hat{Y}_t(t_0)\big) \equiv \begin{cases} m_L(t_0) & \text{for all } t \in (t_0-\varepsilon, t_0) \text{ for some } \varepsilon > 0, \\ n - d_{t_0}^+ & \text{for all } t \in (t_0, t_0+\varepsilon) \text{ for some } \varepsilon > 0, \end{cases} \tag{6.17}$$

$$\mu^*\big(Y(t_0), \hat{Y}_t(t_0)\big) \equiv \begin{cases} n - d_{t_0}^- & \text{for all } t \in (t_0-\varepsilon, t_0) \text{ for some } \varepsilon > 0, \\ m_R(t_0) & \text{for all } t \in (t_0, t_0+\varepsilon) \text{ for some } \varepsilon > 0. \end{cases} \tag{6.18}$$

Equations (6.17) and (6.18) show that the values of the comparative index $\mu\big(Y(t_0), \hat{Y}_t(t_0)\big)$ for t in a right neighborhood of t_0 and of the dual comparative index $\mu^*\big(Y(t_0), \hat{Y}_t(t_0)\big)$ for t in a left neighborhood of t_0 do not depend on the choice of the conjoined basis (X, U). Hence, the limits in (6.14) and (6.15) also do not depend on the choice of (X, U).

Remark 6.7. Formula (6.16) together with Proposition 4.3 implies that if $t_0, t \in [a, b]$ are such that $t_0 < t$ and a conjoined basis (X, U) has constant kernel on $(t_0, t]$, then

$$\mu^*\big(Y(t_0), \hat{Y}_t(t_0)\big) = \operatorname{rank} X(t) - \operatorname{rank} X(t_0), \quad \mu\big(Y(t_0), \hat{Y}_t(t_0)\big) = \operatorname{rank} \hat{X}_t(t_0). \tag{6.19}$$

Similarly, if $t < t_0$ and (X, U) has constant kernel on $[t, t_0)$, then formula (6.16) and Proposition 4.4 yield the equalities

$$\mu\big(Y(t_0), \hat{Y}_t(t_0)\big) = \operatorname{rank} X(t) - \operatorname{rank} X(t_0), \quad \mu^*\big(Y(t_0), \hat{Y}_t(t_0)\big) = \operatorname{rank} \hat{X}_t(t_0). \tag{6.20}$$

The expressions in (6.19) and (6.20) can be utilized for an easier evaluation of the limits in (6.14) and (6.15).

7. Examples and concluding remarks

We conclude this paper by several examples, which illustrate our new theory, and by remarks about the results of this paper, related topics, and future research directions. First we consider a scalar controllable system (H) from [31, Example 7.1].

Example 7.1. Let $n = 1$, $[a, b] = [0, 2]$, $A(t) \equiv 0$, $B(t) = 1 + t^2$, and $C(t) = -2/(1 + t^2)^2$. Since $B(t) > 0$, it follows that system (H) is completely controllable on the interval $[0, 2]$. The principal solutions of (H) at the points $a = 0$ and $b = 2$ are

$$\big(\hat{X}_a(t), \hat{U}_a(t)\big) = \big(t,\ 1/(1 + t^2)\big), \quad \big(\hat{X}_b(t), \hat{U}_b(t)\big) = \big(2t^2 - 3t - 2,\ (4t - 3)/(1 + t^2)\big).$$

For their numbers of left and right proper focal points we then have

$$\widehat{m}_{La}(0, 2] = 0, \quad \widehat{m}_{Ra}[0, 2) = 1, \quad \widehat{m}_{Lb}(0, 2] = 1, \quad \widehat{m}_{Rb}[0, 2) = 0,$$

because $(\hat{X}_a, \hat{U}_a)$ has the only right proper focal point at $t = 0$ and $(\hat{X}_b, \hat{U}_b)$ has the only left proper focal point at $t = 2$. Since $\operatorname{rank} \hat{X}_a(2) = 1 = \operatorname{rank} \hat{X}_b(0)$, we can see that

$$\widehat{m}_{Lb}(0,2] = 1 = \widehat{m}_{La}(0,2] + \operatorname{rank} \hat{X}_a(2), \quad \widehat{m}_{Ra}[0,2) = 1 = \widehat{m}_{Rb}[0,2) + \operatorname{rank} \hat{X}_b(0),$$

as we claim in (5.14) and (5.15) in Theorem 5.3. At the same time we have

$$\widehat{m}_{Ra}[0,2) = 1 = \widehat{m}_{Lb}(0,2], \quad \widehat{m}_{La}(0,2] = 0 = \widehat{m}_{Rb}[0,2),$$

as we claim in (5.16) in Corollary 5.4. Moreover, by (5.18) and (5.19) in Theorem 5.6 every conjoined basis (X,U) of (H) satisfies

$$0 \le m_L(0,2] \le 1, \quad 0 \le m_R[0,2) \le 1. \tag{7.1}$$

For example, consider the conjoined bases

$$\big(X(t),\, U(t)\big) = \big(-t^2+2t+1,\, (-2t+2)/(1+t^2)\big),\ \big(\tilde{X}(t),\, \tilde{U}(t)\big) = \big(-t^2+1,\, -2t/(1+t^2)\big),$$

which of course satisfy (7.1), since $m_L(0,2] = 0 = m_R[0,2)$ and $\widetilde{m}_L(0,2] = 1 = \widetilde{m}_R[0,2)$. In the latter case the left and right proper focal point of $(\tilde{X},\tilde{U})$ is at $t=1$. For the differences of the numbers of the left/right proper focal points of (X,U) and $(\tilde{X},\tilde{U})$ we then have

$$\begin{aligned}
&\big|\, m_L(0,2] - \widetilde{m}_L(0,2] \,\big| = 1, \quad \big|\, m_L(0,2] - \widetilde{m}_R[0,2) \,\big| = 1, \quad \big|\, m_L(0,2] - m_R[0,2) \,\big| = 0 \le 1, \\
&\big|\, m_R[0,2) - \widetilde{m}_R[0,2) \,\big| = 1, \quad \big|\, m_R[0,2) - \widetilde{m}_L(0,2] \,\big| = 1, \quad \big|\, \widetilde{m}_L(0,2] - \widetilde{m}_R[0,2) \,\big| = 0 \le 1,
\end{aligned}$$

which illustrate the estimates (5.24)–(5.27) in Corollary 5.8. Also, since in this case we have $\operatorname{rank} X(0) = 1 = \operatorname{rank} X(2)$ and $\operatorname{rank} \tilde{X}(0) = 1 = \operatorname{rank} \tilde{X}(2)$, the equalities

$$\begin{aligned}
m_L(0,2] + \operatorname{rank} X(2) &= 1 = m_R[0,2) + \operatorname{rank} X(0), \\
\widetilde{m}_L(0,2] + \operatorname{rank} \tilde{X}(2) &= 2 = \widetilde{m}_R[0,2) + \operatorname{rank} \tilde{X}(0)
\end{aligned}$$

illustrate the validity of (5.1) in Theorem 5.1. Concerning the comparative index $\mu\big(Y(t), \tilde{Y}(t)\big)$ and the dual comparative index $\mu^*\big(Y(t), \tilde{Y}(t)\big)$, we have by (2.3)–(2.4) that $W \equiv -2$, $\mathcal{M}(t) \equiv 0$, $V(t) \equiv 1$, and $\mathcal{P}(t) = 2(t^2-1)/(-t^2+2t+1)$. Therefore, $\operatorname{rank} \mathcal{M}(t) \equiv 0$ and

$$\mu\big(Y(t), \tilde{Y}(t)\big) = \operatorname{ind} \mathcal{P}(t) = \operatorname{ind}(t^2-1) = \begin{cases} 1, & t \in [0,1), \\ 0, & t \in [1,2], \end{cases}$$

$$\mu^*\big(Y(t), \tilde{Y}(t)\big) = \operatorname{ind}[-\mathcal{P}(t)] = \operatorname{ind}(1-t^2) = \begin{cases} 0, & t \in [0,1], \\ 1, & t \in (1,2]. \end{cases}$$

Consequently, we obtain

$$\begin{aligned}
m_L(0,2] - \widetilde{m}_L(0,2] &= -1 = \mu\big(Y(2), \tilde{Y}(2)\big) - \mu\big(Y(0), \tilde{Y}(0)\big), \\
m_R[0,2) - \widetilde{m}_R[0,2) &= -1 = \mu^*\big(Y(0), \tilde{Y}(0)\big) - \mu^*\big(Y(2), \tilde{Y}(2)\big),
\end{aligned}$$

which illustrate equalities (4.6) and (4.7) in Theorem 4.1.

In our second example we consider an abnormal system (H) and, in addition to the Sturmian separation theorems illustrated in Example 7.1, we also analyze the continuity and limit properties of the comparative index. This example is motivated by [35, Example 3.4].

Example 7.2. Let $n=2$, $[a,b]=[\frac{\pi}{2},\frac{7\pi}{2}]$, $A(t)\equiv 0$, $C(t)\equiv -I$, and

$$B(t):=\begin{cases}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, & t\in[\frac{\pi}{2},\pi)\cup[2\pi,3\pi),\\[2mm] \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}, & t\in[\pi,2\pi)\cup[3\pi,\frac{7\pi}{2}].\end{cases}$$

If $J\subseteq[\frac{\pi}{2},\frac{7\pi}{2}]$ is a given interval, then the order of abnormality $d(J)=1$ if the interval J does not contain the points π, 2π, 3π in its interior, and otherwise $d(J)=0$. In particular, $d[\frac{\pi}{2},\frac{7\pi}{2}]=0$. The principal solution of (H) at $a=\frac{\pi}{2}$ has the form

$$\big(\hat{X}_a(t),\hat{U}_a(t)\big)=\begin{cases}\left(\begin{pmatrix}-\cos t & 0\\ 0 & 0\end{pmatrix},\begin{pmatrix}\sin t & 0\\ 0 & 1\end{pmatrix}\right), & t\in[\frac{\pi}{2},\pi],\\[2mm] \left(\begin{pmatrix}1 & 0\\ 0 & -\sin t\end{pmatrix},\begin{pmatrix}\pi-t & 0\\ 0 & -\cos t\end{pmatrix}\right), & t\in[\pi,2\pi],\\[2mm] \left(\begin{pmatrix}\cos t-\pi\sin t & 0\\ 0 & 0\end{pmatrix},\begin{pmatrix}-\sin t-\pi\cos t & 0\\ 0 & -1\end{pmatrix}\right), & t\in[2\pi,3\pi],\\[2mm] \left(\begin{pmatrix}-1 & 0\\ 0 & \sin t\end{pmatrix},\begin{pmatrix}t-2\pi & 0\\ 0 & \cos t\end{pmatrix}\right), & t\in[3\pi,\frac{7\pi}{2}],\end{cases}$$

and the principal solution of (H) at $b=\frac{7\pi}{2}$ has the form

$$\big(\hat{X}_b(t),\hat{U}_b(t)\big)=\begin{cases}\left(\begin{pmatrix}\sin t & 0\\ 0 & 1\end{pmatrix},\begin{pmatrix}\cos t & 0\\ 0 & 2\pi-t\end{pmatrix}\right), & t\in[\frac{\pi}{2},\pi],\\[2mm] \left(\begin{pmatrix}0 & 0\\ 0 & -\cos t-\pi\sin t\end{pmatrix},\begin{pmatrix}-1 & 0\\ 0 & \sin t-\pi\cos t\end{pmatrix}\right), & t\in[\pi,2\pi],\\[2mm] \left(\begin{pmatrix}-\sin t & 0\\ 0 & -1\end{pmatrix},\begin{pmatrix}-\cos t & 0\\ 0 & t-3\pi\end{pmatrix}\right), & t\in[2\pi,3\pi],\\[2mm] \left(\begin{pmatrix}0 & 0\\ 0 & \cos t\end{pmatrix},\begin{pmatrix}1 & 0\\ 0 & -\sin t\end{pmatrix}\right), & t\in[3\pi,\frac{7\pi}{2}].\end{cases}$$

For the analysis of the proper focal points of $(\hat{X}_a,\hat{U}_a)$ and $(\hat{X}_b,\hat{U}_b)$ as well as of some other conjoined bases of (H) we introduce the special points $\tau_1,\dots,\tau_6\in(\frac{\pi}{2},\frac{7\pi}{2})$ defined as

$$\begin{aligned}\tau_1 &:= \operatorname{arccotg}(-2\pi)\approx 0.95\pi, & \tau_2 &:= \pi+\operatorname{arccotg}\pi\approx 1.1\pi,\\ \tau_3 &:= \pi+\operatorname{arccotg}(-\pi)\approx 1.9\pi, & \tau_4 &:= 2\pi+\operatorname{arccotg}\pi\approx 2.1\pi,\\ \tau_5 &:= 2\pi+\operatorname{arccotg}(-\pi)\approx 2.9\pi, & \tau_6 &:= 3\pi+\operatorname{arccotg}(2\pi)\approx 3.05\pi.\end{aligned}$$

These points are located according to the inequalities

$$a = \tfrac{\pi}{2} < \tau_1 < \pi < \tau_2 < \tfrac{3\pi}{2} < \tau_3 < 2\pi < \tau_4 < \tfrac{5\pi}{2} < \tau_5 < 3\pi < \tau_6 < b = \tfrac{7\pi}{2}.$$

The numbers and locations of the left and right proper focal points of the principal solutions $(\hat{X}_a, \hat{U}_a)$ and $(\hat{X}_b, \hat{U}_b)$ in the intervals $(\frac{\pi}{2}, \frac{7\pi}{2}]$ and $[\frac{\pi}{2}, \frac{7\pi}{2})$ are

$$\widehat{m}_{La}(\tfrac{\pi}{2}, \tfrac{7\pi}{2}] = 2 \ \text{(located at } 2\pi, \tau_4\text{)}, \qquad \widehat{m}_{Ra}[\tfrac{\pi}{2}, \tfrac{7\pi}{2}) = 4 \ \text{(located at } \tfrac{\pi}{2}, \pi, \tau_4, 3\pi\text{)},$$
$$\widehat{m}_{Lb}(\tfrac{\pi}{2}, \tfrac{7\pi}{2}] = 4 \ \text{(located at } \pi, \tau_3, 3\pi, \tfrac{7\pi}{2}\text{)}, \qquad \widehat{m}_{Rb}[\tfrac{\pi}{2}, \tfrac{7\pi}{2}) = 2 \ \text{(located at } \tau_3, 2\pi\text{)},$$

and the multiplicity of each left and right proper focal point is 1. Since we have in this case $\operatorname{rank} \hat{X}_a(\frac{7\pi}{2}) = 2 = \operatorname{rank} \hat{X}_b(\frac{\pi}{2})$, the equalities

$$\widehat{m}_{Lb}(\tfrac{\pi}{2}, \tfrac{7\pi}{2}] = 4 = \widehat{m}_{La}(\tfrac{\pi}{2}, \tfrac{7\pi}{2}] + \operatorname{rank} \hat{X}_a(\tfrac{7\pi}{2}),$$
$$\widehat{m}_{Ra}[\tfrac{\pi}{2}, \tfrac{7\pi}{2}) = 4 = \widehat{m}_{Rb}[\tfrac{\pi}{2}, \tfrac{7\pi}{2}) + \operatorname{rank} \hat{X}_b(\tfrac{\pi}{2}),$$
$$\widehat{m}_{Ra}[\tfrac{\pi}{2}, \tfrac{7\pi}{2}) = 4 = \widehat{m}_{Lb}(\tfrac{\pi}{2}, \tfrac{7\pi}{2}], \quad \widehat{m}_{La}(\tfrac{\pi}{2}, \tfrac{7\pi}{2}] = 2 = \widehat{m}_{Rb}[\tfrac{\pi}{2}, \tfrac{7\pi}{2})$$

illustrate the validity of Theorem 5.3 and Corollary 5.4. By Theorem 5.6 we conclude that for every conjoined basis (X, U) of (H) we have

$$2 \le m_L(\tfrac{\pi}{2}, \tfrac{7\pi}{2}] \le 4, \quad 2 \le m_R[\tfrac{\pi}{2}, \tfrac{7\pi}{2}) \le 4. \tag{7.2}$$

We demonstrate the validity of (7.2) with the conjoined basis (X, U) defined by

$$\big(X(t), U(t)\big) = \begin{cases} \left(\begin{pmatrix} \cos t & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -\sin t & 0 \\ 0 & -t \end{pmatrix}\right), & t \in [\frac{\pi}{2}, \pi], \\ \left(\begin{pmatrix} -1 & 0 \\ 0 & \pi \sin t - \cos t \end{pmatrix}, \begin{pmatrix} t - \pi & 0 \\ 0 & \pi \cos t + \sin t \end{pmatrix}\right), & t \in [\pi, 2\pi], \\ \left(\begin{pmatrix} \pi \sin t - \cos t & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} \pi \cos t + \sin t & 0 \\ 0 & t - \pi \end{pmatrix}\right), & t \in [2\pi, 3\pi], \\ \left(\begin{pmatrix} 1 & 0 \\ 0 & \cos t - 2\pi \sin t \end{pmatrix}, \begin{pmatrix} 2\pi - t & 0 \\ 0 & -\sin t - 2\pi \cos t \end{pmatrix}\right), & t \in [3\pi, \frac{7\pi}{2}], \end{cases}$$

and the conjoined basis $(\tilde{X}, \tilde{U})$ defined by

$$\big(\tilde{X}(t), \tilde{U}(t)\big) = \begin{cases} \left(\begin{pmatrix} 2\pi \sin t + \cos t & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2\pi \cos t - \sin t & 0 \\ 0 & 2\pi - t \end{pmatrix}\right), & t \in [\frac{\pi}{2}, \pi], \\ \left(\begin{pmatrix} -1 & 0 \\ 0 & -\cos t - \pi \sin t \end{pmatrix}, \begin{pmatrix} t - 3\pi & 0 \\ 0 & \sin t - \pi \cos t \end{pmatrix}\right), & t \in [\pi, 2\pi], \\ \left(\begin{pmatrix} -\cos t - \pi \sin t & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} \sin t - \pi \cos t & 0 \\ 0 & t - 3\pi \end{pmatrix}\right), & t \in [2\pi, 3\pi], \\ \left(\begin{pmatrix} 1 & 0 \\ 0 & \cos t \end{pmatrix}, \begin{pmatrix} 4\pi - t & 0 \\ 0 & -\sin t \end{pmatrix}\right), & t \in [3\pi, \frac{7\pi}{2}]. \end{cases}$$

The numbers and locations of the left and right proper focal points of (X, U) and $(\tilde{X}, \tilde{U})$ are

$$m_L(\tfrac{\pi}{2}, \tfrac{7\pi}{2}] = 3 \ \text{(located at } \tau_2, \tau_4, \tau_6), \qquad m_R[\tfrac{\pi}{2}, \tfrac{7\pi}{2}) = 4 \ \text{(located at } \tfrac{\pi}{2}, \tau_2, \tau_4, \tau_6),$$
$$\widetilde{m}_L(\tfrac{\pi}{2}, \tfrac{7\pi}{2}] = 4 \ \text{(located at } \tau_1, \tau_3, \tau_5, \tfrac{7\pi}{2}), \qquad \widetilde{m}_R[\tfrac{\pi}{2}, \tfrac{7\pi}{2}) = 3 \ \text{(located at } \tau_1, \tau_3, \tau_5),$$

and the multiplicity of each left and right proper focal point is again 1. Since in this case $\operatorname{rank} \hat{X}_a(\frac{7\pi}{2}) = 2 = \operatorname{rank} \hat{X}_b(\frac{\pi}{2})$, the inequalities in Corollary 5.8 are then illustrated by

$$\begin{aligned}
&\left| m_L(\tfrac{\pi}{2}, \tfrac{7\pi}{2}] - \widetilde{m}_L(\tfrac{\pi}{2}, \tfrac{7\pi}{2}] \right| = 1 \le 2, \quad && \left| m_R[\tfrac{\pi}{2}, \tfrac{7\pi}{2}) - \widetilde{m}_R[\tfrac{\pi}{2}, \tfrac{7\pi}{2}) \right| = 1 \le 2,\\
&\left| m_L(\tfrac{\pi}{2}, \tfrac{7\pi}{2}] - \widetilde{m}_R[\tfrac{\pi}{2}, \tfrac{7\pi}{2}) \right| = 0 \le 2, \quad && \left| m_R[\tfrac{\pi}{2}, \tfrac{7\pi}{2}) - \widetilde{m}_L(\tfrac{\pi}{2}, \tfrac{7\pi}{2}] \right| = 0 \le 2,\\
&\left| m_L(\tfrac{\pi}{2}, \tfrac{7\pi}{2}] - m_R[\tfrac{\pi}{2}, \tfrac{7\pi}{2}) \right| = 1 \le 2, \quad && \left| \widetilde{m}_L(\tfrac{\pi}{2}, \tfrac{7\pi}{2}] - \widetilde{m}_R[\tfrac{\pi}{2}, \tfrac{7\pi}{2}) \right| = 1 \le 2.
\end{aligned}$$

Also, since $\operatorname{rank} X(\frac{\pi}{2}) = 1 = \operatorname{rank} \tilde{X}(\frac{7\pi}{2})$ and $\operatorname{rank} X(\frac{7\pi}{2}) = 2 = \operatorname{rank} \tilde{X}(\frac{\pi}{2})$, the equalities

$$\begin{aligned}
m_L(\tfrac{\pi}{2}, \tfrac{7\pi}{2}] + \operatorname{rank} X(\tfrac{7\pi}{2}) &= 5 = m_R[\tfrac{\pi}{2}, \tfrac{7\pi}{2}) + \operatorname{rank} X(\tfrac{\pi}{2}),\\
\widetilde{m}_L(\tfrac{\pi}{2}, \tfrac{7\pi}{2}] + \operatorname{rank} \tilde{X}(\tfrac{7\pi}{2}) &= 5 = \widetilde{m}_R[\tfrac{\pi}{2}, \tfrac{7\pi}{2}) + \operatorname{rank} \tilde{X}(\tfrac{\pi}{2})
\end{aligned}$$

illustrate the validity of Theorem 5.1.

Next we calculate the comparative index $\mu\big(Y(t), \tilde{Y}(t)\big)$ and the dual comparative index $\mu^*\big(Y(t), \tilde{Y}(t)\big)$. The constant Wronskian of (X, U) and $(\tilde{X}, \tilde{U})$ is $W = 2\pi I$. Therefore, by (2.4) we have $\mathcal{M}(t) = 2\pi\,[I - X^{\dagger}(t)\,X(t)]$. This shows that $\mathcal{M}(t)$ is symmetric and $\operatorname{rank} \mathcal{M}(t) = \operatorname{def} X(t)$. This implies that $V(t) = X^{\dagger}(t)\,X(t) = P(t)$ and $\mathcal{P}(t) = 2\pi\, X^{\dagger}(t)\,\tilde{X}(t)\,P(t)$. Since the matrices $X(t)$ and $\tilde{X}(t)$ are diagonal, they commute and $\operatorname{ind} \mathcal{P}(t) = \operatorname{ind} X(t)\,\tilde{X}(t)$. Therefore, by (2.3) we have in this case the expressions

$$\mu\big(Y(t), \tilde{Y}(t)\big) = \operatorname{def} X(t) + \operatorname{ind} X(t)\,\tilde{X}(t), \qquad \mu^*\big(Y(t), \tilde{Y}(t)\big) = \operatorname{def} X(t) + \operatorname{ind}[-X(t)\,\tilde{X}(t)].$$

This yields after straightforward calculations that

$$\mu(t) := \mu\big(Y(t), \tilde{Y}(t)\big) = \begin{cases} 1, & t \in [\frac{\pi}{2}, \tau_1) \cup [\tau_2, \tau_3) \cup [\tau_4, \tau_5) \cup [\tau_6, \frac{7\pi}{2}),\\ 0, & t \in [\tau_1, \tau_2) \cup [\tau_3, \tau_4) \cup [\tau_5, \tau_6) \cup \{\frac{7\pi}{2}\}, \end{cases}$$

$$\mu^*(t) := \mu^*\big(Y(t), \tilde{Y}(t)\big) = \begin{cases} 2, & t \in \{\frac{\pi}{2}\} \cup (\tau_1, \tau_2] \cup (\tau_3, \tau_4] \cup (\tau_5, \tau_6],\\ 1, & t \in (\frac{\pi}{2}, \tau_1] \cup (\tau_2, \tau_3] \cup (\tau_4, \tau_5] \cup (\tau_6, \frac{7\pi}{2}]. \end{cases}$$

We can see that $\mu(t)$ and $\mu^*(t)$ are piecewise constant on $[\frac{\pi}{2}, \frac{7\pi}{2}]$ and that $\mu(t)$ is right-continuous on $[\frac{\pi}{2}, \frac{7\pi}{2})$ and $\mu^*(t)$ is left-continuous on $(\frac{\pi}{2}, \frac{7\pi}{2}]$, as we claim in Theorem 6.1. The discontinuity points of $\mu(t)$ are located at τ_1–τ_6 and at $\frac{7\pi}{2}$, and the discontinuity points of $\mu^*(t)$ are located at $\frac{\pi}{2}$ and at τ_1–τ_6. Moreover, according to (6.1) and (6.2) the jumps in the values of $\mu(t)$ and $\mu^*(t)$ at each discontinuity point t_0 satisfy

$$\mu(t_0) - \lim_{t\to t_0^-} \mu(t) = m_L(t_0) - \widetilde{m}_L(t_0), \qquad \mu^*(t_0) - \lim_{t\to t_0^+} \mu^*(t) = m_R(t_0) - \widetilde{m}_R(t_0). \tag{7.3}$$

For example, the differences in (7.3) are equal to 1 when $t_0 \in \{\tau_2, \tau_4, \tau_6\}$ and they are equal to -1 when $t_0 \in \{\tau_1, \tau_3, \tau_5\}$. Also, Theorem 4.1 is illustrated by the equalities

$$m_L(\tfrac{\pi}{2}, \tfrac{7\pi}{2}] - \widetilde{m}_L(\tfrac{\pi}{2}, \tfrac{7\pi}{2}] = -1 = \mu(\tfrac{7\pi}{2}) - \mu(\tfrac{\pi}{2}),$$
$$m_R[\tfrac{\pi}{2}, \tfrac{7\pi}{2}) - \widetilde{m}_R[\tfrac{\pi}{2}, \tfrac{7\pi}{2}) = 1 = \mu^*(\tfrac{\pi}{2}) - \mu^*(\tfrac{7\pi}{2}).$$

Next we will illustrate the validity of Theorems 6.3 and 6.5 at some given t_0, e.g., at the point $t_0 = \tau_2 \in [\pi, 2\pi]$. For this purpose we need to calculate the value of the principal solution $(\hat{X}_t, \hat{U}_t)$ at the point τ_2 for t in some neighborhood of τ_2. We can directly verify that

$$\big(\hat{X}_t(s), \hat{U}_t(s)\big) = \left(\begin{pmatrix} 0 & 0 \\ 0 & \sin(s-t) \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \cos(s-t) \end{pmatrix} \right), \quad s, t \in [\pi, 2\pi]. \tag{7.4}$$

The conjoined basis (X, U) is invertible (and hence has constant kernel) on the intervals $[\pi, \tau_2)$ and $(\tau_2, 2\pi]$ and it has a left and right proper focal point at τ_2, with $\operatorname{rank} X(\tau_2) = 1$. Therefore, we obtain from Remark 6.7 that

$$\mu^*\big(Y(\tau_2), \hat{Y}_t(\tau_2)\big) = 1, \quad \mu\big(Y(\tau_2), \hat{Y}_t(\tau_2)\big) = \operatorname{rank} \hat{X}_t(\tau_2), \quad t \in (\tau_2, 2\pi], \tag{7.5}$$

$$\mu\big(Y(\tau_2), \hat{Y}_t(\tau_2)\big) = 1, \quad \mu^*\big(Y(\tau_2), \hat{Y}_t(\tau_2)\big) = \operatorname{rank} \hat{X}_t(\tau_2), \quad t \in [\pi, \tau_2). \tag{7.6}$$

From (7.4) we get that $\operatorname{rank} \hat{X}_t(\tau_2) = 1$ for all $t \in [\pi, 2\pi]$, $t \neq \tau_2$. Since $d[t, \tau_2] = 1$ for $t \in [\pi, \tau_2)$ and $d[\tau_2, t] = 1$ for $t \in (\tau_2, 2\pi]$, it follows by (3.17) and (3.18) that $d^{\pm}_{\tau_2} = 1$. Then we obtain from (7.5) and (7.6) that

$$\lim_{t \to \tau_2^-} \mu\big(Y(\tau_2), \hat{Y}_t(\tau_2)\big) = 1 = m_L(\tau_2), \qquad \lim_{t \to \tau_2^+} \mu\big(Y(\tau_2), \hat{Y}_t(\tau_2)\big) = 1 = n - d^+_{\tau_2},$$
$$\lim_{t \to \tau_2^-} \mu^*\big(Y(\tau_2), \hat{Y}_t(\tau_2)\big) = 1 = n - d^-_{\tau_2}, \qquad \lim_{t \to \tau_2^+} \mu^*\big(Y(\tau_2), \hat{Y}_t(\tau_2)\big) = 1 = m_R(\tau_2).$$

These one-sided limits coincide with those in Theorems 6.3 and 6.5, compare also with Remark 6.6.

Remark 7.3. The results in this paper (in particular, in Theorems 4.1, 5.2, 5.6, 5.9, 6.1, 6.3 and in Corollaries 5.8, 5.10) are new even for a completely controllable system (H). In this case the left and right proper focal points of (X, U) coincide, since the matrix $X(t)$ is invertible on $[a, b]$ except at finitely many isolated points.

Remark 7.4. Some results in this paper about the numbers of focal points of conjoined bases of linear Hamiltonian system (H) can be regarded as continuous analogs of discrete time results for conjoined bases of a symplectic difference system. More precisely, formulas (4.6) and (4.7) correspond to [13, Corollary 3.1 and Equation (3.4)], formula (5.1) corresponds to [13, Equation (3.5)], formulas (5.14) and (5.15) correspond to [7, Equations (4.27a)–(4.27b)], formula (5.16) corresponds to [13, Lemma 3.3] and [7, Equations (4.29a)–(4.29b)], inequality (5.31) corresponds to [3, Theorem 3.2], and formulas (6.8) and (6.9) correspond to [13, Corollary 3.1]. In this respect the discrete time theory motivates the development in the continuous time theory, which is the opposite direction compared to the traditional approach. On the other hand, it is easy to see that the new estimates in Theorems 5.6 and 5.9 and in Corollaries 5.8 and 5.10 can now be derived also in the discrete time setting for symplectic systems.

Remark 7.5. During the preparation of the final version of this paper in June 2016, we were notified about the paper [15] by J. Elyseeva. Her paper deals with comparison and separation theorems for left proper focal points in $(a, b]$ of two conjoined bases of two different linear Hamiltonian systems of the form (H) satisfying the Sturm majorant condition. Hence, in some sense her results are more general than ours in this paper. More specifically, formula (4.6) in our Theorem 4.1, the statement (i) in our Theorem 6.1, and equality (6.8) in our Theorem 6.3 are contained in [15] as Theorems 2.2, 2.3, and Lemma 3.1. We would like to emphasize that the results in the present paper were derived independently of [15] and, to our knowledge, in about the same time. Also, our proofs use different techniques and they are more straightforward, as we deal with one system (H) only. In addition, we derive our results for the right proper focal points as well, and relate them with those for the left proper focal points.

Remark 7.6. The results in this paper open new directions in the oscillation theory of linear Hamiltonian systems as well as self-adjoint Sturm–Liouville differential equations. For example, singular Sturmian type theorems for a controllable system (H) on unbounded intervals $[a, \infty)$, $(-\infty, b]$, or $(-\infty, \infty)$ are proven in [11]. We are convinced that it is now possible to generalize and complete the results in [11] to uncontrollable systems (H) by using the comparative index. We also believe that the comparative index will lead to solving the fundamental questions about possible distribution and locations of (left and right proper) focal points of conjoined bases of (H) in $[a, b]$. These topics will be discussed in our subsequent work.

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APPENDIX D

Paper by Šepitka & Šimon Hilscher (JDE 2019)

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Singular Sturmian separation theorems on unbounded intervals for linear Hamiltonian systems ☆

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Abstract

In this paper we develop new fundamental results in the Sturmian theory for nonoscillatory linear Hamiltonian systems on an unbounded interval. We introduce a new concept of a multiplicity of a focal point at infinity for conjoined bases and, based on this notion, we prove singular Sturmian separation theorems on an unbounded interval. The main results are formulated in terms of the (minimal) principal solutions at both endpoints of the considered interval, and include exact formulas as well as optimal estimates for the numbers of proper focal points of one or two conjoined bases. As a natural tool we use the comparative index, which was recently implemented into the theory of linear Hamiltonian systems by the authors and independently by J. Elyseeva. Throughout the paper we do not assume any controllability condition on the system. Our results turn out to be new even in the completely controllable case.

MSC: 34C10

Keywords: Sturmian separation theorem; Linear Hamiltonian system; Proper focal point; Minimal principal solution; Antiprincipal solution; Comparative index

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1. Introduction

Let $n \in \mathbb{N}$ be a given dimension. In this paper we consider the linear Hamiltonian system

$$x' = A(t)\,x + B(t)\,u, \quad u' = C(t)\,x - A^T(t)\,u, \quad t \in \mathcal{I}, \tag{H}$$

where $\mathcal{I} \subseteq \mathbb{R}$ is a fixed interval (not necessarily compact) and $A, B, C : \mathcal{I} \to \mathbb{R}^{n\times n}$ are given piecewise continuous matrix-valued functions on $\mathcal{I}$ such that $B(t)$ and $C(t)$ are symmetric and the Legendre condition holds, i.e.,

$$B(t) \geq 0 \quad \text{for all } t \in \mathcal{I}. \tag{1.1}$$

The purpose of the paper is to develop new fundamental results in the Sturmian theory of a nonoscillatory system (H) on the interval $\mathcal{I} = [a, \infty)$ or $\mathcal{I} = (-\infty, b]$, in particular to derive the Sturmian separation theorems concerning the numbers of focal points in $\mathcal{I}$ of conjoined bases of (H). We refer to Section 2 for the definitions of a conjoined basis and the nonoscillation of (H). We show that the known Sturmian separation theorems on a compact interval $\mathcal{I} = [a, b]$ can be extended to the unbounded interval $\mathcal{I} = [a, \infty)$ or $\mathcal{I} = (-\infty, b]$. The main ingredients in this extension are the new concept of a multiplicity of a focal point at infinity for conjoined bases of (H), which we introduce in this paper, and using the minimal principal solution of (H) at infinity from [25] as the reference solution for counting the focal points. As a natural tool, which connects these two concepts, we use the comparative index from [9,10], which was recently implemented into the theory of linear Hamiltonian systems by the authors in [30] and independently by Elyseeva in [12,13]. We note that the first applications of the comparative index in the continuous time theory were derived in [11, Section 3].

It is known in [19, Theorem 3] or in [14, Proof of Lemma 3.6(a)] that under (1.1) every conjoined basis (X, U) of (H) has the kernel of $X(t)$ piecewise constant on $\mathcal{I}$, i.e., the kernel of $X(t)$ changes finitely many times in any compact subinterval of $\mathcal{I}$. In this case we say that (X, U) has a *left proper focal point* at $t_0 \in \mathcal{I}$ if $\operatorname{Ker} X(t_0^-) \subsetneqq \operatorname{Ker} X(t_0)$ with the multiplicity

$$m_L(t_0) := \operatorname{def} X(t_0) - \operatorname{def} X(t_0^-), \tag{1.2}$$

and a *right proper focal point* at $t_0 \in \mathcal{I}$ if $\operatorname{Ker} X(t_0^+) \subsetneqq \operatorname{Ker} X(t_0)$ with the multiplicity

$$m_R(t_0) := \operatorname{def} X(t_0) - \operatorname{def} X(t_0^+), \tag{1.3}$$

see [20,34]. The notations $\operatorname{Ker} X(t_0^\pm)$, $\operatorname{def} X(t_0^\pm)$, and later $\operatorname{rank} X(t_0^\pm)$ represent the one-sided limits at t_0 of the piecewise constant quantities $\operatorname{Ker} X(t)$, $\operatorname{def} X(t) := \dim \operatorname{Ker} X(t)$, and $\operatorname{rank} X(t)$.

In the historical development of the Sturmian theory for system (H) on a compact interval $\mathcal{I} = [a, b]$ the principal solutions at the points a and b play a fundamental role, see [21, Corollary 1, p. 336], [20, Corollary 4.8], [32, Theorems 1.4–1.5], and recently [30, Theorem 5.6]. We recall that the *principal solution* $(\hat{X}_s, \hat{U}_s)$ of (H) at the point $s \in [a, b]$ is defined as the solution of (H) starting with the initial conditions

$$\hat{X}_s(s) = 0, \quad \hat{U}_s(s) = I. \tag{1.4}$$

For future reference in the paper we state the following result from [30, Theorem 5.6]. We emphasize that the focal points are always counted including their multiplicities. A more detailed statement is presented in Proposition 2.2.

Proposition 1.1. *Assume that* (1.1) *holds with* $\mathcal{I} = [a, b]$. *If the principal solution* $(\hat{X}_a, \hat{U}_a)$ *of* (H) *has* m *left proper focal points in* $(a, b]$, *then any other conjoined basis of* (H) *has at least* m *and at most* $m + \operatorname{rank} \hat{X}_a(b)$ *left proper focal points in* $(a, b]$. *Similarly, if the principal solution* $(\hat{X}_b, \hat{U}_b)$ *of* (H) *has* m *right proper focal points in* $[a, b)$, *then any other conjoined basis of* (H) *has at least* m *and at most* $m + \operatorname{rank} \hat{X}_b(a)$ *right proper focal points in* $[a, b)$.

We note that $\operatorname{rank} \hat{X}_a(b) = \operatorname{rank} \hat{X}_b(a)$, since this quantity is equal to the rank of the (constant) Wronskian of the two solutions $(\hat{X}_a, \hat{U}_a)$ and $(\hat{X}_b, \hat{U}_b)$.

The above result in Proposition 1.1 holds for a general system (H) without any controllability assumption. Recall that system (H) is completely controllable on $\mathcal{I}$ if the trivial solution $\big(x(t), u(t)\big) \equiv (0, 0)$ is the only solution of (H), for which $x(t) \equiv 0$ on a nondegenerate subinterval of $\mathcal{I}$. When $\mathcal{I} = [a, \infty)$ resp. $\mathcal{I} = (-\infty, \infty)$ and system (H) is completely controllable on $\mathcal{I}$, then Došlý and Kratz proved in [8, Theorem 1 and Corollary 1] the following results.

Proposition 1.2. *Assume that* (1.1) *holds with* $\mathcal{I} = [a, \infty)$, *system* (H) *is completely controllable on* $[a, \infty)$ *and nonoscillatory at* ∞. *If the principal solution of* (H) *at infinity has* m *focal points in* $[a, \infty)$, *then any other conjoined basis of* (H) *has at least* m *focal points in* $[a, \infty)$.

Proposition 1.3. *Assume that* (1.1) *holds with* $\mathcal{I} = (-\infty, \infty)$, *system* (H) *is completely controllable on* $(-\infty, \infty)$ *and nonoscillatory at* $\pm\infty$. *Then the principal solutions of* (H) *at infinity and minus infinity have the same number of focal points in* $(-\infty, \infty)$.

For a completely controllable system (H) on $[a, \infty)$ we know by [18, Theorem 4.1.3] that $\operatorname{Ker} X(t_0^{\pm}) = \{0\}$ for every point $t_0 \in [a, \infty)$. This means that the notions of left and right proper focal points in (1.2) and (1.3) coincide, i.e., $m_L(t_0) = m_R(t_0)$, and the corresponding multiplicity is

$$m(t_0) := \operatorname{def} X(t_0). \tag{1.5}$$

Recall that the principal solution $(\hat{X}_\infty, \hat{U}_\infty)$ of (H) at infinity is defined as a conjoined basis of (H), for which $\hat{X}_\infty(t)$ is invertible on some interval $[\alpha, \infty)$ and

$$\lim_{t\to\infty} \bigg(\int_\alpha^t \hat{X}_\infty^{-1}(s)\, B(s)\, \hat{X}_\infty^{T-1}(s)\, \mathrm{d}s \bigg)^{-1} = 0.$$

According to [21, Theorem VII.3.3] or [5, Theorem 3, p. 43] or [16, Section XI.10.5(i)–(ii)], this solution exists and is unique up to a constant right nonsingular multiple when system (H) is nonoscillatory and completely controllable on $[a, \infty)$. However, the questions regarding the validity of the estimates in Proposition 1.1 on unbounded intervals (i.e., for $b = \infty$), as well as removing the complete controllability assumption in Propositions 1.2 and 1.3 remained open.

In this paper we provide a solution to both of the two above problems. We show that in the absence of the controllability assumption the minimal principal solution of (H) at infinity from

[25,26] should be used as the reference solution for counting the (left and right proper) focal points. According to [25, Definition 7.1], this solution is defined as a conjoined basis $(\hat{X}_\infty, \hat{U}_\infty)$ of (H), for which the kernel of $\hat{X}_\infty(t)$ is constant on some interval $[\alpha, \infty)$ and

$$\lim_{t\to\infty} \left(\int_\alpha^t \hat{X}_\infty^\dagger(s)\, B(s)\, \hat{X}_\infty^{\dagger\, T}(s)\, \mathrm{d}s \right)^\dagger = 0, \tag{1.6}$$

where the dagger denotes the Moore–Penrose pseudoinverse [2–4]. Similarly to the controllable case, the minimal principal solution of (H) at infinity exists and is unique up to a constant right nonsingular multiple when the system (H) is nonoscillatory, see [25, Theorems 7.2 and 7.6] and Proposition 2.9.

The second main ingredient of the present paper is concerned with the definition of the multiplicity of a focal point at infinity (Definition 3.1) for conjoined bases of (H). This is a completely new notion in the theory of differential equations (it is new even in the controllable case), which is related to a unified view on the principal solutions of (H) at a finite point and at infinity. In [31, Theorem 5.8] we proved that these two types of solutions coincide, that is, the principal solution at t_0 is in fact the (left and right) minimal principal solution of (H) at t_0 in a sense parallel to (1.6). Then, motivated by the result in [31, Theorem 6.1], we define the multiplicity of a focal point of (X, U) at infinity as the difference of the defect of its associated T-matrix $T_{\alpha,\infty}$ and the order of abnormality of system (H) on $[\alpha, \infty)$. The matrix $T_{\alpha,\infty}$ is defined by

$$T_{\alpha,\infty} := \lim_{t\to\infty} S_\alpha^\dagger(t), \quad S_\alpha(t) := \int_\alpha^t X^\dagger(s)\, B(s)\, X^{\dagger\, T}(s)\, \mathrm{d}s, \quad t \in [\alpha, \infty), \tag{1.7}$$

where $\alpha \in [a, \infty)$ is such that the kernel of $X(t)$ is constant on $[\alpha, \infty)$. Using this new concept we prove (Theorem 5.7) that the results in Proposition 1.1 extend naturally to unbounded intervals, when the multiplicities of the left proper focal points are counted in the interval $(a, \infty]$. In particular, the multiplicity of the (left) focal point at infinity should be included.

The results in Proposition 1.1 are essentially based on using the *comparative index*, which was introduced by Elyseeva in [9,10], to express the difference of the numbers of left proper focal points of two conjoined bases (X, U) and $(\tilde{X}, \tilde{U})$ of (H) in $(a, b]$. Similarly, the *dual comparative index* is used for the difference of the numbers of right proper focal points of (X, U) and $(\tilde{X}, \tilde{U})$ in $[a, b)$, see Subsection 2.2. As a main tool for proving our new Sturmian separation theorems we derive (Theorem 5.1) extensions of the above mentioned formulas to unbounded intervals $(a, \infty]$ or $[a, \infty)$. In this approach we also apply the limit characterization of the minimal principal solution of (H) at infinity in different genera of conjoined bases from [26, Corollary 5.5], as well as a newly derived characterization of antiprincipal solutions of (H) at infinity in different genera of conjoined bases (Theorems A.1 and A.4 in the appendix). Our new results also include optimal estimates for numbers of focal points of one or two conjoined bases (Theorems 5.7 and 5.10 and Corollaries 5.8 and 5.13), as well as limit properties of the comparative index at infinity (Theorem 6.1) and its relationship with the multiplicities of focal points at infinity (Theorem 6.4). As an application of the main Sturmian separation theorems for system (H) we derive a singular version of the Sturmian separation theorem for the second order Sturm–Liouville differential equations (Remark 7.1) and discuss the corresponding notion of disconjugacy on the unbounded interval $[a, \infty]$ (Theorem 7.2). We note that all the presented results extend naturally to the unbounded intervals of the form $(-\infty, \infty]$, $[-\infty, \infty)$, or $(-\infty, \infty)$ (Remark 8.1). We also wish to emphasize that all the results are new even for the completely controllable system (H).

In conclusion, we believe that this paper provides a new perspective in understanding the Sturmian theory of linear Hamiltonian systems and Sturm–Liouville differential equations on unbounded intervals. We are also convinced that these results will stimulate further development in the oscillation theory of differential equations in general.

2. Conjoined bases and their properties

In this section we present an overview of the properties of conjoined bases of (H) in the general possibly uncontrollable case. We also recall the definition of a comparative index, the order of abnormality of (H), the nonoscillation and genera of conjoined bases of (H), and the principal and antiprincipal solutions of (H) at infinity.

2.1. Conjoined bases

We adopt a usual convention that $2n \times n$ matrix-valued solutions of (H) will be denoted by the capital letters, typically by Y, $\tilde{Y}$, $\bar{Y}$, $\hat{Y}$, etc. In this case we split the solutions into two $n \times n$ blocks denoted by X and U (preserving the notation in Y), i.e.,

$$Y(t) = \begin{pmatrix} X(t) \\ U(t) \end{pmatrix}, \quad \tilde{Y}(t) = \begin{pmatrix} \tilde{X}(t) \\ \tilde{U}(t) \end{pmatrix}, \quad \bar{Y}(t) = \begin{pmatrix} \bar{X}(t) \\ \bar{U}(t) \end{pmatrix} \tag{2.1}$$

for generic conjoined bases of (H), or

$$\hat{Y}_s(t) = \begin{pmatrix} \hat{X}_s(t) \\ \hat{U}_s(t) \end{pmatrix}, \quad \hat{Y}_\infty(t) = \begin{pmatrix} \hat{X}_\infty(t) \\ \hat{U}_\infty(t) \end{pmatrix}, \quad \hat{Y}_{-\infty}(t) = \begin{pmatrix} \hat{X}_{-\infty}(t) \\ \hat{U}_{-\infty}(t) \end{pmatrix} \tag{2.2}$$

for the (minimal) principal solutions of (H) at the point $s \in \mathcal{I}$, resp. at plus/minus infinity.

A solution Y of (H) is a *conjoined basis* if $X^T(t)\, U(t)$ is symmetric and $\operatorname{rank} Y(t) = n$ for some (and hence for any) $t \in \mathcal{I}$. The principal solution $\hat{Y}_s$ for $s \in \mathcal{I}$, which is given by the initial conditions (1.4), is an example of such a conjoined basis. For two solutions Y and $\bar{Y}$ of (H) their *Wronskian*

$$W(Y, \bar{Y}) := Y^T(t)\, \mathcal{J}\, \bar{Y}(t) = X^T(t)\, \bar{U}(t) - U^T(t)\, \bar{X}(t) \tag{2.3}$$

is constant on $\mathcal{I}$, since its derivative is zero throughout $\mathcal{I}$. Any conjoined basis Y forms a one half of a symplectic fundamental matrix $\Phi(t)$ of (H), i.e.,

$$\Phi(t) = \begin{pmatrix} Y(t) & \bar{Y}(t) \end{pmatrix}, \quad \Phi^T(t)\, \mathcal{J}\, \Phi(t) = \mathcal{J}, \quad \mathcal{J} := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{2.4}$$

In this case the conjoined bases Y and $\bar{Y}$ are normalized in a sense that their Wronskian is $W(Y, \bar{Y}) = I$. This can be equivalently formulated as

$$X\bar{U}^T - \bar{X}U^T = I, \quad X\bar{X}^T = \bar{X}X^T, \quad U\bar{U}^T = \bar{U}U^T, \tag{2.5}$$

saying that $\Phi^{-1}(t) = -\mathcal{J}\,\Phi^T(t)\,\mathcal{J}$ and $\Phi(t)\,\Phi^{-1}(t) = I$. We note that for any conjoined bases Y, $\bar{Y}$, $\tilde{Y}$ of (H) such that $W(Y, \bar{Y}) = I$ the $n \times n$ matrix

$$W(\tilde{Y}, Y)\,[W(\tilde{Y}, \bar{Y})]^T \quad \text{is symmetric.} \tag{2.6}$$

The proof of (2.6) follows from the properties in (2.5).

Next we consider constant real $2n \times n$ matrices Y_i, $i \in \{1, 2, 3, 4\}$, and derive additional properties regarding their Wronskian type matrices $W(Y_i, Y_j) = Y_i^T \mathcal{J}\, Y_j$. In applications of these properties in Section 4 the matrices Y_i will be the values of conjoined bases of (H) at some fixed point $t_0 \in \mathcal{I}$.

Proposition 2.1. *Let Y_1, Y_2, Y_3, Y_4 be real constant $2n \times n$ matrices. Then*

$$W(Y_1, Y_2)\,W(Y_3, Y_4) - W(Y_1, Y_3)\,W(Y_2, Y_4) = Y_1^T \mathcal{J} \begin{pmatrix} Y_2 & Y_3 \end{pmatrix} \mathcal{J} \begin{pmatrix} Y_2 & Y_3 \end{pmatrix}^T \mathcal{J}\, Y_4. \tag{2.7}$$

In particular, if $W(Y_2, Y_2) = 0 = W(Y_3, Y_3)$ and $W(Y_3, Y_2) = I$, then

$$W(Y_1, Y_2)\,W(Y_3, Y_4) - W(Y_1, Y_3)\,W(Y_2, Y_4) = W(Y_1, Y_4). \tag{2.8}$$

Proof. Identity (2.7) follows by direct calculations by using that $Y_2\,Y_3^T - Y_3\,Y_2^T = \mathcal{Y}\mathcal{J}\mathcal{Y}^T$ with the $2n \times 2n$ matrix $\mathcal{Y} := \begin{pmatrix} Y_2 & Y_3 \end{pmatrix}$. If in addition $W(Y_2, Y_2) = 0 = W(Y_3, Y_3)$ and $W(Y_3, Y_2) = I$, then the matrix $\mathcal{Y}$ satisfies $\mathcal{Y}^T \mathcal{J}\, \mathcal{Y} = -\mathcal{J}$, and hence also $\mathcal{Y}\,\mathcal{J}\,\mathcal{Y}^T = -\mathcal{J}$. Therefore, identity (2.8) follows from (2.7). $\square$

If Y is a conjoined basis of (H), then we use for simplicity the terminology *kernel of* Y, *image of* Y, and *rank of* Y for the quantities $\operatorname{Ker} X$, $\operatorname{Im} X$, and $\operatorname{rank} X$, respectively. In this context the property that $Y(t)$ has a constant kernel on some interval $\mathcal{I}_0 \subseteq \mathcal{I}$ means that the kernel of $X(t)$ is constant on $\mathcal{I}_0$.

2.2. *Focal points and comparative index on compact interval*

For a conjoined basis Y of (H) the multiplicities of its left and right proper focal points are defined, under (1.1), by formulas (1.2) and (1.3). For the interval $\mathcal{I} = [a, b]$ we denote by

$$m_L(a, b] := \text{ the number of left proper focal points of } Y \text{ in } (a, b], \tag{2.9}$$

$$m_R[a, b) := \text{ the number of right proper focal points of } Y \text{ in } [a, b). \tag{2.10}$$

In the same spirit as in (2.9) and (2.10) we will use the notations $\widetilde{m}_L(a, b]$, $\widetilde{m}_R[a, b)$ and $\widehat{m}_{Ls}(a, b]$, $\widehat{m}_{Rs}[a, b)$ for the numbers of left and right proper focal points of a conjoined basis $\tilde{Y}$ and of the principal solution $\hat{Y}_s$ in the given interval, typically with $s \in \{a, b, t\}$ or later in Section 5 with $s = \pm\infty$. The following regular Sturmian separation theorems were derived in [30, Section 5], compare also with Proposition 1.1.

Proposition 2.2. *Assume that* (1.1) *holds with $\mathcal{I} = [a, b]$. Then for any conjoined basis Y of* (H) *and the principal solutions $\hat{Y}_a$ and $\hat{Y}_b$ we have*

$$m_L(a,b] + \operatorname{rank} X(b) = m_R[a,b) + \operatorname{rank} X(a), \tag{2.11}$$

$$\widehat{m}_{Lb}(a,b] = \widehat{m}_{La}(a,b] + \operatorname{rank} \hat{X}_a(b), \quad \widehat{m}_{Ra}[a,b) = \widehat{m}_{Rb}[a,b) + \operatorname{rank} \hat{X}_b(a), \tag{2.12}$$

$$\widehat{m}_{Ra}[a,b) = \widehat{m}_{Lb}(a,b], \quad \widehat{m}_{La}(a,b] = \widehat{m}_{Rb}[a,b), \tag{2.13}$$

$$\widehat{m}_{La}(a,b] \le m_L(a,b] \le \widehat{m}_{Lb}(a,b], \quad \widehat{m}_{Rb}[a,b) \le m_R[a,b) \le \widehat{m}_{Ra}[a,b). \tag{2.14}$$

The identities and estimates in Proposition 2.2 are based on the exact formulas

$$m_L(a,b] - \widetilde{m}_L(a,b] = \mu\big(Y(b), \tilde{Y}(b)\big) - \mu\big(Y(a), \tilde{Y}(a)\big), \tag{2.15}$$

$$m_R[a,b) - \widetilde{m}_R[a,b) = \mu^*\big(Y(a), \tilde{Y}(a)\big) - \mu^*\big(Y(b), \tilde{Y}(b)\big), \tag{2.16}$$

$$m_L(a,b] = \widehat{m}_{La}(a,b] + \mu\big(Y(b), \hat{Y}_a(b)\big), \tag{2.17}$$

$$m_R[a,b) = \widehat{m}_{Rb}[a,b) + \mu^*\big(Y(a), \hat{Y}_b(a)\big), \tag{2.18}$$

which involve the comparative index $\mu(Y, \tilde{Y})$ and the dual comparative index $\mu^*(Y, \tilde{Y})$, see [30, Theorem 4.1, Equation (5.28)] and also [12, Theorem 2.3]. In addition, the multiplicities in (1.2) and (1.3) are related with the comparative index by the formulas, see [30, Theorem 6.3] or [12, Lemma 3.1],

$$m_L(t_0) = \lim_{t \to t_0^-} \mu\big(Y(t_0), \hat{Y}_t(t_0)\big), \quad m_R(t_0) = \lim_{t \to t_0^+} \mu^*\big(Y(t_0), \hat{Y}_t(t_0)\big). \tag{2.19}$$

More precisely, following [9, Definition 2.1] or [10, Definition 2.1] we define for two real constant $2n \times n$ matrices Y and $\tilde{Y}$ such that

$$Y^T \mathcal{J}\, Y = 0, \quad \tilde{Y}^T \mathcal{J}\, \tilde{Y} = 0, \quad \operatorname{rank} Y = n = \operatorname{rank} \tilde{Y}, \quad W := Y^T \mathcal{J}\, \tilde{Y} \tag{2.20}$$

their *comparative index* $\mu(Y, \tilde{Y})$ and the *dual comparative index* $\mu^*(Y, \tilde{Y})$ as the numbers

$$\mu(Y, \tilde{Y}) := \operatorname{rank} \mathcal{M} + \operatorname{ind} \mathcal{P}, \quad \mu^*(Y, \tilde{Y}) := \operatorname{rank} \mathcal{M} + \operatorname{ind}(-\mathcal{P}), \tag{2.21}$$

where $\mathcal{M}$ and $\mathcal{P}$ are the $n \times n$ matrices

$$\mathcal{M} := (I - X^\dagger X)\, W, \quad \mathcal{P} := V\, W^T X^\dagger \tilde{X}\, V, \quad V := I - \mathcal{M}^\dagger \mathcal{M}. \tag{2.22}$$

Here Y and $\tilde{Y}$ are partitioned into $n \times n$ blocks according to the notation in (2.1). We note that the matrix V is the orthogonal projector onto $\operatorname{Ker} \mathcal{M}$ and the matrix $\mathcal{P}$ is symmetric, see [10, Theorem 2.1]. The quantity $\operatorname{ind} \mathcal{P}$ denotes the index of $\mathcal{P}$, i.e., the number of its negative eigenvalues, and obviously $\operatorname{ind}(-\mathcal{P}) = \operatorname{rank} \mathcal{P} - \operatorname{ind} \mathcal{P}$. Among the algebraic properties of the comparative index in [10, Section 2], see also [30, Section 2] for their overview, we mention

$$\mu(Y, \tilde{Y}) + \operatorname{rank} X = \mu^*(\tilde{Y}, Y) + \operatorname{rank} \tilde{X}, \tag{2.23}$$

$$\mu(Y, \tilde{Y}) + \mu(\tilde{Y}, Y) = \operatorname{rank} W = \mu^*(Y, \tilde{Y}) + \mu^*(\tilde{Y}, Y), \tag{2.24}$$

$$\mu(Y, E) = 0 = \mu^*(Y, E), \quad \mu(E, Y) = \operatorname{rank} X = \mu^*(E, Y), \quad E := (0,\ I)^T. \tag{2.25}$$

The following transformation formulas were proven in [10, Theorem 2.2].

Proposition 2.3. *For arbitrary symplectic $2n \times 2n$ matrices $\mathcal{S}$, Φ, and $\tilde{\Phi}$ we have*

$$\mu(\mathcal{S}\,\Phi E, \mathcal{S}E) - \mu(\mathcal{S}\,\tilde{\Phi}E, \mathcal{S}E) = \mu(\mathcal{S}\,\Phi E, \mathcal{S}\,\tilde{\Phi}E) - \mu(\Phi E, \tilde{\Phi}E), \tag{2.26}$$

$$\mu^*(\mathcal{S}\,\Phi E, \mathcal{S}E) - \mu^*(\mathcal{S}\,\tilde{\Phi}E, \mathcal{S}E) = \mu^*(\mathcal{S}\,\Phi E, \mathcal{S}\,\tilde{\Phi}E) - \mu^*(\Phi E, \tilde{\Phi}E). \tag{2.27}$$

In addition, by using (2.20)–(2.22) we can check easily the identities

$$\mu(-Y, \tilde{Y}) = \mu(Y, -\tilde{Y}) = \mu(Y, \tilde{Y}), \quad \mu^*(-Y, \tilde{Y}) = \mu^*(Y, -\tilde{Y}) = \mu^*(Y, \tilde{Y}). \tag{2.28}$$

2.3. Order of abnormality

We denote by $d[t,\infty)$ the dimension of the space of vector solutions $(x \equiv 0, u)$ of (H) on the interval $[t,\infty)$. This number is called an *order of abnormality* of system (H) on $[t,\infty)$. Then $0 \le d[t,\infty) \le n$ and the integer-valued function $d[t,\infty)$ is nondecreasing, piecewise constant, and right-continuous on $[a,\infty)$. Therefore, there exists the *maximal order of abnormality* d_∞, which satisfies

$$d_\infty := \lim_{t\to\infty} d[t,\infty) = \max_{t\in[a,\infty)} d[t,\infty), \quad 0 \le d_\infty \le n. \tag{2.29}$$

Obviously, for a completely controllable system (H) we have $d[t,\infty) = d_\infty = 0$ for all $t \in [a,\infty)$. In a similar way we define for $\mathcal{I} = (-\infty, b]$ the quantity

$$d_{-\infty} := \lim_{t\to-\infty} d(-\infty,t] = \max_{t\in(-\infty,b]} d(-\infty,t], \quad 0 \le d_{-\infty} \le n, \tag{2.30}$$

compare with [31, Equations (2.16)–(2.17)].

2.4. Nonoscillation of system (H)

Let $\mathcal{I} = [a,\infty)$. We say that a conjoined basis Y of (H) is *nonoscillatory* (at ∞) if there exists $\alpha \in [a,\infty)$ such that Y has no left proper focal points in the interval (α,∞). In this case we may assume without loss of generality that the point α is such that $\operatorname{Ker} X(t)$ is constant on $[\alpha,\infty)$. By [33, Theorem 2.2] we have the following.

Proposition 2.4. *Assume that* (1.1) *holds with $\mathcal{I} = [a,\infty)$. Then there exists a nonoscillatory conjoined basis of* (H) *at ∞ if and only if every conjoined basis of* (H) *is nonoscillatory at ∞.*

Based on this result we say that system (H) is *nonoscillatory* if one (and hence every) conjoined basis of (H) is nonoscillatory (at ∞). In the opposite case we say that system (H) is oscillatory (at ∞). In a similar way we define for $\mathcal{I} = (-\infty, b]$ the nonoscillation of system (H) at $-\infty$ in terms of the nonexistence of right proper focal points of a conjoined basis Y in the interval $(-\infty,\beta)$ for some $\beta \in (-\infty, b]$, or equivalently by the property of having constant kernel of $X(t)$ on $(-\infty,\beta]$. The opposite situation defines the notion of an oscillatory system (H) at $-\infty$. We note that we can use either the left proper focal points or the right proper focal points to define the oscillation of a conjoined basis Y at $\pm\infty$, since by (2.11) we have

$$\left|m_L(a,b] - m_R[a,b)\right| = \left|\operatorname{rank} X(a) - \operatorname{rank} X(b)\right| \le n.$$

Let $\mathcal{I} = [a, \infty)$ again and let system (H) be nonoscillatory at ∞. Let Y be a conjoined basis of (H) with constant kernel on $[\alpha, \infty)$. We define the constant orthogonal projector P onto $\operatorname{Im} X^T(t) = [\operatorname{Ker} X(t)]^\perp$ and the orthogonal projector $R(t)$ onto $\operatorname{Im} X(t)$ on $[\alpha, \infty)$ by

$$P := X^\dagger(t)\, X(t), \quad R(t) := X(t)\, X^\dagger(t), \quad t \in [\alpha, \infty). \tag{2.31}$$

In this case Y has constant rank r on $[\alpha, \infty)$ with, see [25, Equation (5.13)],

$$r := \operatorname{rank} X(t) = \operatorname{rank} P = \operatorname{rank} R(t), \quad t \in [\alpha, \infty), \quad n - d[\alpha, \infty) \le r \le n. \tag{2.32}$$

The matrix $X^\dagger(t)$ is then piecewise continuously differentiable on $[\alpha, \infty)$ by [4, Theorems 10.5.1 and 10.5.3]. This yields that the associated matrix $S_\alpha(t)$ in (1.7) is well defined on $[\alpha, \infty)$ and, under (1.1), it is symmetric, nonnegative definite, and piecewise continuously differentiable on $[\alpha, \infty)$. Moreover, the set $\operatorname{Im} S_\alpha(t)$ is nondecreasing and hence eventually constant with $\operatorname{Im} S_\alpha(t) \subseteq \operatorname{Im} P$ on $[\alpha, \infty)$, see [25, Theorem 4.2]. This implies that the orthogonal projector $P_{\mathcal{S}_\alpha}(t)$ onto $\operatorname{Im} S_\alpha(t)$ is eventually constant and we write

$$P_{\mathcal{S}_\alpha}(t) := S_\alpha^\dagger(t)\, S_\alpha(t) = S_\alpha(t)\, S_\alpha^\dagger(t), \quad t \in [\alpha, \infty), \quad P_{\mathcal{S}_\alpha\infty} := P_{\mathcal{S}_\alpha}(t), \quad t \to \infty. \tag{2.33}$$

In addition, by [25, Theorem 5.2 and Remarks 5.3 and 6.2(ii)] we have the properties

$$\left.\begin{aligned} \operatorname{Im} S_\alpha(t) = \operatorname{Im} P_{\mathcal{S}_\alpha}(t) \subseteq \operatorname{Im} P_{\mathcal{S}_\alpha\infty} \subseteq \operatorname{Im} P, \quad t \in [\alpha, \infty), \\ \operatorname{Im} T_{\alpha,\infty} \subseteq \operatorname{Im} P_{\mathcal{S}_\alpha\infty}, \quad \operatorname{rank} P_{\mathcal{S}_\alpha\infty} = n - d[\alpha, \infty). \end{aligned}\right\} \tag{2.34}$$

The function $S_\alpha(t)$ is closely related with a certain class of conjoined bases of (H), which are normalized with Y. More precisely, in [25, Theorem 4.4] we proved that for a given conjoined basis Y with constant kernel on $[\alpha, \infty)$ there exists a conjoined basis $\bar{Y}$ of (H) such that Y and $\bar{Y}$ are normalized, i.e., (2.5) holds, and

$$X^\dagger(\alpha)\, \bar{X}(\alpha) = 0. \tag{2.35}$$

Moreover, by [25, Remark 4.5(ii)] the matrix $\bar{X}(t)$ is uniquely determined by Y on $[\alpha, \infty)$, as well as the matrices

$$\bar{X}(t)\, P = X(t)\, S_\alpha(t), \quad \bar{U}(t)\, P = U(t)\, S_\alpha(t) + X^{\dagger T}(t) + U(t)\, (I - P)\, \bar{X}^T(t)\, X^{\dagger T}(t) \tag{2.36}$$

are uniquely determined by Y on $[\alpha, \infty)$.

The following result from [25, Theorem 4.6] shows that under a certain condition the conjoined bases of (H) with constant kernel on $[\alpha, \infty)$ are mutually representable.

Proposition 2.5. *Assume that (1.1) holds with $\mathcal{I} = [a, \infty)$. Let Y_1 and Y_2 be conjoined bases of (H) with constant kernels on $[\alpha, \infty)$ and let P_1 and P_2 be the projectors defined in (2.31) through the functions $X_1(t)$ and $X_2(t)$, respectively. Let Y_2 be expressed in terms of Y_1 via matrices M_1, N_1 and let Y_1 be expressed in terms of Y_2 via matrices M_2, N_2, that is,*

$$\begin{pmatrix} X_2 \\ U_2 \end{pmatrix} = \begin{pmatrix} X_1 & \bar{X}_1 \\ U_1 & \bar{U}_1 \end{pmatrix} \begin{pmatrix} M_1 \\ N_1 \end{pmatrix}, \quad \begin{pmatrix} X_1 \\ U_1 \end{pmatrix} = \begin{pmatrix} X_2 & \bar{X}_2 \\ U_2 & \bar{U}_2 \end{pmatrix} \begin{pmatrix} M_2 \\ N_2 \end{pmatrix} \quad \textit{on } [\alpha, \infty), \tag{2.37}$$

where $\bar{Y}_1$ and $\bar{Y}_2$ are conjoined bases of (H) *satisfying* (2.5) *and* (2.35) *with regard to the conjoined bases Y_1 and Y_2. If $\operatorname{Im} X_1(\alpha) = \operatorname{Im} X_2(\alpha)$, then*

(i) *$M_1^T N_1$ and $M_2^T N_2$ are symmetric and $N_1 + N_2^T = 0$,*
(ii) *M_1 and M_2 are nonsingular, $M_1 M_2 = M_2 M_1 = I$, and $P_2 M_2 = (P_1 M_1)^\dagger$,*
(iii) *$\operatorname{Im} N_1 \subseteq \operatorname{Im} P_1$ and $\operatorname{Im} N_2 \subseteq \operatorname{Im} P_2$.*

Moreover, the matrices M_1, N_1 do not depend on the choice of $\bar{Y}_1$, and the matrices M_2, N_2 do not depend on the choice of $\bar{Y}_2$, namely

$$N_1 = W(Y_1, Y_2), \quad N_2 = W(Y_2, Y_1) = -N_1^T, \quad M_1 = -W(\bar{Y}_1, Y_2), \quad M_2 = -W(\bar{Y}_2, Y_1). \tag{2.38}$$

The first equality in (2.36) applied to the conjoined bases Y_1 and Y_2 allows to rewrite expressions (2.37) into the form

$$X_{3-i}(t) = X_i(t)\,[P_i\, M_i + S_{i\alpha}(t)\, N_i], \quad t \in [\alpha, \infty), \quad i \in \{1, 2\}, \tag{2.39}$$

where $S_{1\alpha}(t)$ and $S_{2\alpha}(t)$ are associated with Y_1 and Y_2 through (1.7). Hence, under the assumptions of Proposition 2.5 we have $\operatorname{Im} X_1(t) = \operatorname{Im} X_2(t)$ on $[\alpha, \infty)$, that is, the conjoined bases Y_1 and Y_2 have eventually the same image.

2.5. Genera of conjoined bases

Let system (H) be nonoscillatory at ∞. As in [26, Definition 6.3 and Remark 6.4] we define a *genus* $\mathcal{G}^\infty$ of conjoined bases of (H) as an equivalence class of all conjoined bases of (H) which have eventually the same image. In this case we define the rank of $\mathcal{G}^\infty$ as the eventual rank of some (or any) conjoined basis $Y \in \mathcal{G}^\infty$, see also [27, Remark 6.4]. Moreover, from (2.32) and (2.29) it follows that $n - d_\infty \le \operatorname{rank} \mathcal{G}^\infty \le n$.

Conjoined bases Y of (H), which have eventually the smallest possible rank $n - d_\infty$ according to (2.32), form the unique *minimal genus* $\mathcal{G}^\infty_{\min}$. Similarly, conjoined bases having the largest possible rank n form the unique *maximal genus* $\mathcal{G}^\infty_{\max}$. That is, for $Y \in \mathcal{G}^\infty_{\max}$ the matrix $X(t)$ is eventually invertible, see also [26, Remarks 7.14 and 7.15].

In [28, Section 4] we introduced an ordering $\mathcal{G}^\infty_1 \preceq \mathcal{G}^\infty_2$ between two genera of conjoined bases by the inclusion between the images of their representing conjoined bases, i.e., eventually $\operatorname{Im} X_1(t) \subseteq \operatorname{Im} X_2(t)$ holds for $Y_1 \in \mathcal{G}^\infty_1$ and $Y_2 \in \mathcal{G}^\infty_2$. In particular, the results in [28, Theorem 4.8 and Remark 4.7] say that the set Γ^∞ of all genera of conjoined bases of (H) forms a complete lattice, where $\mathcal{G}^\infty_1 \wedge \mathcal{G}^\infty_2$ and $\mathcal{G}^\infty_1 \vee \mathcal{G}^\infty_2$ denote the genera represented by conjoined bases having their image eventually equal to $\operatorname{Im} X_1(t) \cap \operatorname{Im} X_2(t)$ and $\operatorname{Im} X_1(t) + \operatorname{Im} X_2(t)$, respectively. In this case the two genera $\mathcal{G}^\infty_{\min}$ and $\mathcal{G}^\infty_{\max}$ are the smallest and the largest elements of the set Γ^∞ in this ordering.

In a similar way we treat the genera of conjoined bases in the neighborhood of $-\infty$ by using the notation $\mathcal{G}^{-\infty}$, $\mathcal{G}^{-\infty}_{\min}$, $\mathcal{G}^{-\infty}_{\max}$, etc.

Remark 2.6. The theory of genera of conjoined bases of (H) at $\pm\infty$ discussed above as well as in [24,26–28] extends under the Legendre condition (1.1) in a straightforward way to the left and right neighborhoods of any finite point $t_0 \in [a,\infty)$. This is a consequence of the fact that the left and right proper focal points of any conjoined basis of (H) are isolated, by [19, Theorem 3]. In this context we will use the notation $\mathcal{G}(t_0^\pm)$ for the genus corresponding to a conjoined basis Y in a left/right neighborhood of the point t_0, or the notation $\mathcal{G}_{\min}(t_0^\pm)$ and $\mathcal{G}_{\max}(t_0^\pm)$ for the corresponding minimal and maximal genus. This idea is similar to the unification of the theory of principal and antiprincipal solutions at $\pm\infty$ and at a finite point t_0, which was recently developed in [31].

2.6. Relation being contained for conjoined bases

In the following subsection we recall a construction of conjoined bases of (H) with constant kernel through a relation "being contained", see [25, Section 5] for more details. Let Y and Y_* be two conjoined bases of (H) such that Y has constant kernel on $[\alpha,\infty)$. Let P and $P_{\mathcal{S}_\alpha\infty}$ be the associated orthogonal projectors for Y defined in (2.31) and (2.33). We say that Y_* is *contained in* Y on $[\alpha,\infty)$, or that Y *contains* the conjoined basis Y_* on $[\alpha,\infty)$, if there exists an orthogonal projector P_* such that $X_*(t) = X(t)\,P_*$ on $[\alpha,\infty)$ and

$$\operatorname{Im} P_{\mathcal{S}_\alpha\infty} \subseteq \operatorname{Im} P_* \subseteq \operatorname{Im} P. \tag{2.40}$$

It follows that every conjoined basis Y_* of (H), which is contained in a conjoined basis Y of (H) with constant kernel on $[\alpha,\infty)$, has also a constant kernel on $[\alpha,\infty)$ with

$$\operatorname{Ker} X_*(t) = \operatorname{Ker} P_*, \quad X_*^\dagger(t) = X^\dagger(t)\,R_*(t) \quad \text{on } [\alpha,\infty), \tag{2.41}$$

where the matrix $R_*(t)$ is defined in (2.31) through $X_*(t)$. The importance of the relation being contained can be seen from the following results, see [25, Section 5].

Remark 2.7. (i) In [25, Theorem 5.11] we proved that the relation "being contained" preserves the corresponding S-matrices, and hence also the T-matrices. More precisely, if Y is a conjoined basis of (H) with constant kernel on $[\alpha,\infty)$ and $S_\alpha(t)$ is the associated matrix in (1.7), then for any conjoined basis Y_* of (H), which is contained in Y on $[\alpha,\infty)$, its corresponding matrix $S_{*\alpha}(t)$ satisfies the equality $S_{*\alpha}(t) = S_\alpha(t)$ for all $t \in [\alpha,\infty)$, and consequently also $T_{*\alpha,\infty} = T_{\alpha,\infty}$.

(ii) From [25, Remark 5.13] it follows that every conjoined basis of (H) from the minimal genus $\mathcal{G}_{\min}^\infty$, which has constant kernel on $[\alpha,\infty)$, can be constructed from a given conjoined basis Y with constant kernel on $[\alpha,\infty)$ by using the relation "being contained" with the choice of $P_* := P_{\mathcal{S}_\alpha\infty}$. Moreover, as we comment in [26, Remark 7.14] and in Subsection 2.5, for any two conjoined bases Y_1 and Y_2 of (H) belonging to $\mathcal{G}_{\min}^\infty$ there exists $\alpha \in \mathcal{I}$ such that Y_1 and Y_2 have constant kernel on $[\alpha,\infty)$ and the equality $\operatorname{Im} X_1(t) = \operatorname{Im} X_2(t)$ holds on $[\alpha,\infty)$. In particular, if M_1 and N_1 are the associated constant matrices in Proposition 2.5, then the matrices $T_{1\alpha,\infty}$ and $T_{2\alpha,\infty}$ in (1.7) corresponding to Y_1 and Y_2 satisfy

$$T_{2\alpha,\infty} = M_1^T\, T_{1\alpha,\infty}\, M_1 + M_1^T N_1. \tag{2.42}$$

The following result from [28, Remark 4.12] shows that the relation being contained allows to construct conjoined bases from a genus $\mathcal{H}^\infty$ by using conjoined bases from a given genus

$\mathcal{G}^\infty$ satisfying $\mathcal{H}^\infty \preceq \mathcal{G}^\infty$. This construction will be utilized in the proof of Theorem A.4 in the appendix.

Proposition 2.8. *Assume that* (1.1) *holds with* $\mathcal{I} = [a, \infty)$. *Let* Y *be a conjoined basis of* (H) *from a given genus* $\mathcal{G}^\infty$ *and let* $\alpha \in \mathcal{I}$ *be such that* Y *has constant kernel on* $[\alpha, \infty)$. *Moreover, let* P *and* $P_{\mathcal{S}_\alpha\infty}$ *be the associated matrices in* (2.31) *and* (2.33). *Then for every genus* $\mathcal{H}^\infty \preceq \mathcal{G}^\infty$ *there exists a unique orthogonal projector* P_* *satisfying* (2.40) *such that* Y *contains a conjoined basis* Y_* *of* (H) *on* $[\alpha, \infty)$ *with respect to* P_*, *which belongs to the genus* $\mathcal{H}^\infty$.

2.7. Principal and antiprincipal solutions at infinity

Following the discussion about the S-matrices in (1.7) and Subsection 2.4, we observe that for a conjoined basis Y of (H) with constant kernel on $[\alpha, \infty)$ the matrix-valued function $S_\alpha^\dagger(t)$ is nonnegative definite and nonincreasing on $[\alpha, \infty)$. Therefore, the matrix $T_{\alpha,\infty}$ defined in (1.7) exists, it is symmetric and nonnegative definite, and $\operatorname{Im} T_{\alpha,\infty} \subseteq \operatorname{Im} P_{\mathcal{S}_\alpha\infty}$, see [25, Remark 6.2(ii)]. Moreover, we know from [31, Theorem 10.3] that under (1.1) the subspace $\operatorname{Im} T_{\alpha,\infty} \oplus \operatorname{Ker} P_{\mathcal{S}_\alpha\infty}$ and its dimension $d[\alpha, \infty) + \operatorname{rank} T_{\alpha,\infty}$ do not depend on the choice of the point $\alpha \in [a, \infty)$, for which the kernel of Y is constant on $[\alpha, \infty)$. In particular, the estimates

$$d_\infty \le d[\alpha, \infty) + \operatorname{rank} T_{\alpha,\infty} \le n, \tag{2.43}$$

hold, see [31, Remarks 5.2 and 5.4]. When the abnormality of (H) is maximal on $[\alpha, \infty)$, we obtain from (2.43) that $0 \le \operatorname{rank} T_{\alpha,\infty} \le n - d_\infty$, see also [27, Corollary 4.11]. In the two extreme cases in (2.43), i.e., for the values of $\operatorname{rank} T_{\alpha,\infty}$ equal to $d_\infty - d[\alpha, \infty)$ and $n - d[\alpha, \infty)$, we say that Y is a *principal solution of* (H) *at* ∞ and an *antiprincipal solution of* (H) *at* ∞, respectively, see [31, Definition 5.1] and compare with (1.6). It follows from the above discussion that these definitions are correct. The principal solutions of (H) at ∞ will be denoted by $\hat{Y}_\infty$, as we discussed in (2.2).

By [26, Theorem 7.12] and [27, Theorem 5.12] we know that in every genus $\mathcal{G}^\infty$ there exists a principal and an antiprincipal solution of (H) at ∞. In particular, for the case of the minimal genus $\mathcal{G}^\infty_{\min}$ we will use the terminology *minimal principal* and *minimal antiprincipal solution of* (H) *at* ∞, which have their rank equal to $n - d_\infty$. Similarly, for the case of the maximal genus $\mathcal{G}^\infty_{\max}$ we will use the terminology *maximal principal* and *maximal antiprincipal solution of* (H) *at* ∞, which have their rank equal to n. The following result is from [25, Theorems 7.2 and 7.6], compare also with [26, Theorem 7.6] and [27, Theorem 5.8].

Proposition 2.9. *Assume that* (1.1) *holds with* $\mathcal{I} = [a, \infty)$. *System* (H) *is nonoscillatory at* ∞ *if and only if there exists a minimal principal solution* $\hat{Y}_\infty$ *of* (H) *at* ∞. *In this case the solution* $\hat{Y}_\infty$ *is unique up to a constant right nonsingular multiple.*

Remark 2.10. We note that, according to Remark 2.7(i), the property of being a principal or an antiprincipal solution of (H) at ∞ is preserved under the relation being contained for conjoined bases of (H) with constant kernel on the interval $[\alpha, \infty)$.

For the proof of our main result in Theorem 3.3 we will need the following limit property of the minimal principal solution $\hat{Y}_\infty$ of (H) at ∞, see [28, Corollary 5.5] and [29, Proposition 1].

Proposition 2.11. *Assume that* (1.1) *holds with* $\mathcal{I} = [a, \infty)$ *and system* (H) *is nonoscillatory at* ∞. *Let* $\hat{Y}_\infty$ *be the minimal principal solution of* (H) *at* ∞ *and let* $\alpha \in [a, \infty)$ *be such that* $\hat{Y}_\infty$ *has constant kernel on the interval* $[\alpha, \infty)$. *Then every conjoined basis* $\bar{Y}$ *of* (H) *satisfying* (2.5) *and* (2.35) *with respect to* $\hat{Y}_\infty$ *is a maximal antiprincipal solution at* ∞ *and*

$$\lim_{t\to\infty} \bar{X}^{-1}(t)\, \hat{X}_\infty(t) = 0.$$

In a similar way we treat the principal and antiprincipal solutions of (H) at $-\infty$. More precisely, under (1.1) with $\mathcal{I} = (-\infty, b]$ and for a nonoscillatory system (H) at $-\infty$ we use the notation $\hat{Y}_{-\infty}$ for the (unique) minimal principal solution of (H) at $-\infty$. If Y is a conjoined basis of (H) with constant kernel on $(-\infty, \beta]$ for some $\beta \in (-\infty, b]$ and if $S_\beta(t)$ is defined in (1.7) with $\alpha := \beta$ for $t \in (-\infty, \beta]$, then the corresponding T-matrix will be denoted by

$$T_{\beta,\,-\infty} := \lim_{t\to-\infty} S_\beta^\dagger(t), \quad T_{\beta,\,-\infty} \le 0, \quad d_{-\infty} \le d(-\infty, \beta] + \operatorname{rank} T_{\beta,\,-\infty} \le n, \tag{2.44}$$

compare with (2.43) and with [31, Section 5].

3. Multiplicity of focal point at infinity

In this section we introduce one of the key concepts of this paper, which is the multiplicity of the focal point at ∞ for a conjoined basis Y of (H). This notion is motivated by the result in [31, Theorem 6.1 and Remark 6.4], in which we characterized the multiplicity of the left proper focal point at t_0 in (1.2) in terms of the order of abnormality of (H) near t_0 and the rank of the associated T-matrix. More precisely, if Y is a conjoined basis of (H) with constant kernel on the interval $[\alpha, t_0)$, then

$$m_L(t_0) = n - d[\alpha, t_0) - \operatorname{rank} T_{\alpha,\, t_0^-}, \quad T_{\alpha,\, t_0^-} := \lim_{t\to t_0^-} S_\alpha^\dagger(t),$$

while if Y has constant kernel on the interval $(t_0, \alpha]$, then

$$m_R(t_0) = n - d(t_0, \alpha] - \operatorname{rank} T_{\alpha,\, t_0^+}, \quad T_{\alpha,\, t_0^+} := \lim_{t\to t_0^+} S_\alpha^\dagger(t),$$

where $d[\alpha, t_0)$ and $d(t_0, \alpha]$ are the orders of abnormality of (H) on the intervals $[\alpha, t_0)$ and $(t_0, \alpha]$, respectively. Furthermore, we derive an equivalent formula for the multiplicity at ∞ resembling the original definition in (1.2). The results in this section are fundamental for the development of the Sturmian theory of system (H) on the unbounded interval $[a, \infty)$ in Section 5. We note that the presented notion and the results are new even for a completely controllable system (H), see Remark 3.7 below.

Definition 3.1 *(Multiplicity of focal point at* ∞*).* Let $\mathcal{I} = [a, \infty)$ and let Y be a conjoined basis of (H) with constant kernel on the interval $[\alpha, \infty)$ for some $\alpha \in [a, \infty)$. We say that Y has a *(left) proper focal point at* ∞ if $d[\alpha, \infty) + \operatorname{rank} T_{\alpha,\,\infty} < n$ with the multiplicity

$$m_L(\infty) := n - d[\alpha, \infty) - \operatorname{rank} T_{\alpha,\,\infty}, \tag{3.1}$$

where $T_{\alpha,\,\infty}$ is the matrix defined in (1.7) corresponding to Y.

Remark 3.2. (i) In accordance with Subsection 2.7 we note that under (1.1) the number $m_L(\infty)$ defined in (3.1) does not depend on the particular choice of the point $\alpha \in [a,\infty)$, for which the conjoined basis Y has constant kernel on $[\alpha,\infty)$. In particular, the inequalities in (2.43) imply the estimates $0 \le m_L(\infty) \le n - d_\infty$. We also point out that the quantity $m_L(\infty)$ in (3.1) is preserved under the relation being contained for conjoined bases of (H), since the matrix $T_{\alpha,\infty}$ is preserved under this relation, see Subsection 2.6.

(ii) From (3.1) it follows that the conjoined basis Y has no focal point at ∞ if and only if Y is an antiprincipal solution of (H) at ∞. Indeed, the multiplicity $m_L(\infty) = 0$ if and only if $\operatorname{rank} T_{\alpha,\infty} = n - d[\alpha,\infty)$, which is according to Subsection 2.7 the defining property of an antiprincipal solution at ∞. Similarly, the multiplicity $m_L(\infty) = n - d_\infty$ is maximal possible if and only if Y is a principal solution of (H) at ∞, since in this case $\operatorname{rank} T_{\alpha,\infty} = d_\infty - d[\alpha,\infty)$.

In the next result we present a way for computing the multiplicity of the focal point at ∞ in terms of the rank of the genus of a conjoined basis Y near ∞ and the rank of the Wronskian of Y with the minimal principal solution at ∞. This is obviously much simpler for practical calculations than the expression in (3.1). In Remark 3.5 below we shall discuss this result in the relation with the definition of the multiplicity of a left focal point at t_0 in (1.2).

Theorem 3.3. *Assume that (1.1) holds with $\mathcal{I} = [a,\infty)$ and system (H) is nonoscillatory at ∞. Let Y be a conjoined basis of (H) with constant kernel on $[\alpha,\infty) \subseteq \mathcal{I}$ belonging to a genus $\mathcal{G}^\infty$ with the associated matrices P, $P_{\mathcal{S}_\alpha\infty}$, and $T_{\alpha,\infty}$ in (2.31), (2.33), and (1.7). Let $\hat{Y}_\infty$ be the minimal principal solution of (H) at ∞. Then*

$$\operatorname{Im}[W(\hat{Y}_\infty, Y)]^T = \operatorname{Im} T_{\alpha,\infty} \oplus \operatorname{Im}(P - P_{\mathcal{S}_\alpha\infty}), \tag{3.2}$$

$$\operatorname{rank} T_{\alpha,\infty} = \operatorname{rank} W(\hat{Y}_\infty, Y) + n - \operatorname{rank}\mathcal{G}^\infty - d[\alpha,\infty). \tag{3.3}$$

Moreover, the multiplicity of the focal point of Y at ∞ defined in (3.1) satisfies

$$m_L(\infty) = \operatorname{rank}\mathcal{G}^\infty - \operatorname{rank} W(\hat{Y}_\infty, Y). \tag{3.4}$$

Proof. Let α be as in the theorem and choose $\beta \in [\alpha,\infty)$ such that $d[\beta,\infty) = d_\infty$. Denote by $T_{\beta,\infty}$ and $P_{\mathcal{S}_\beta\infty}$ the matrices in (1.7) and (2.33) associated with the conjoined basis Y on the interval $[\beta,\infty)$. Without lost of generality we may assume that the minimal principal solution $\hat{Y}_\infty$ has constant kernel on $[\beta,\infty)$. Moreover, let $\bar{Y}_\beta$ be a conjoined basis of (H) satisfying (2.5) and (2.35) with $Y := \hat{Y}_\infty$ and $\alpha := \beta$. It then follows from Proposition 2.11 that $\bar{Y}_\beta$ is a maximal antiprincipal solution of (H) at ∞ and the equality

$$\lim_{t\to\infty} \bar{X}_\beta^{-1}(t)\,\hat{X}_\infty(t) = 0 \tag{3.5}$$

holds. Following the notation in Proposition 2.5 we represent Y in terms of $\hat{Y}_\infty$ and $\bar{Y}_\beta$ via (constant) matrices $\hat{M}_\infty$ and $\hat{N}_\infty := W(\hat{Y}_\infty, Y)$. That is,

$$\begin{pmatrix} X(t) \\ U(t) \end{pmatrix} = \begin{pmatrix} \hat{X}_\infty(t) & \bar{X}_\beta(t) \\ \hat{U}_\infty(t) & \bar{U}_\beta(t) \end{pmatrix} \begin{pmatrix} \hat{M}_\infty \\ \hat{N}_\infty \end{pmatrix}, \quad t \in [\beta,\infty). \tag{3.6}$$

In particular, the representation in (3.6) implies the formula

$$X(t) = \hat{X}_\infty(t)\,\hat{M}_\infty + \bar{X}_\beta(t)\,\hat{N}_\infty, \quad t \in [\beta, \infty). \tag{3.7}$$

In turn, by combining (3.5) and (3.7) we obtain that

$$\lim_{t\to\infty} \bar{X}_\beta^{-1}(t)\,X(t) \overset{(3.7)}{=} \lim_{t\to\infty} [\bar{X}_\beta^{-1}(t)\,\hat{X}_\infty(t)\,\hat{M}_\infty + \hat{N}_\infty] \overset{(3.5)}{=} \hat{N}_\infty. \tag{3.8}$$

On the other hand, from Corollary A.6 in the appendix with $\alpha := \beta$, $Y := \bar{Y}_\beta$, $\tilde{Y} := Y$, $\tilde{P} := P$, $P_{\tilde{\mathcal{S}}_\alpha\infty} := P_{\mathcal{S}_\beta\infty}$, and $\tilde{T}_{\alpha,\infty} := T_{\beta,\infty}$ it follows that

$$\lim_{t\to\infty} \bar{X}_\beta^{-1}(t)\,X(t) = L \quad \text{with} \quad \operatorname{Im} L^T = \operatorname{Im} T_{\beta,\infty} \oplus \operatorname{Im}(P - P_{\mathcal{S}_\beta\infty}). \tag{3.9}$$

Therefore, by using (3.8) and (3.9) we get the equality $\hat{N}_\infty = L$ and

$$\operatorname{Im} \hat{N}_\infty^T = \operatorname{Im} T_{\beta,\infty} \oplus \operatorname{Im}(P - P_{\mathcal{S}_\beta\infty}). \tag{3.10}$$

Since $\operatorname{Im} T_{t,\infty} \subseteq \operatorname{Im} P_{\mathcal{S}_t\infty} \subseteq \operatorname{Im} P$ for $t \in [\alpha, \infty)$ by (2.34), we know that

$$\begin{aligned} \operatorname{Im} T_{t,\infty} \oplus \operatorname{Im}(P - P_{\mathcal{S}_t\infty}) &= \operatorname{Im} T_{t,\infty} \oplus (\operatorname{Im} P \cap \operatorname{Ker} P_{\mathcal{S}_t\infty}) \\ &= \operatorname{Im} P \cap \big(\operatorname{Im} T_{t,\infty} \oplus \operatorname{Ker} P_{\mathcal{S}_t\infty}\big), \quad t \in [\alpha, \infty). \end{aligned}$$

But since the subspace $\operatorname{Im} T_{t,\infty} \oplus \operatorname{Ker} P_{\mathcal{S}_t\infty}$ does not depend on the choice of $t \in [\alpha, \infty)$ by [31, Theorem 10.3], it follows that the subspace $\operatorname{Im} T_{t,\infty} \oplus \operatorname{Im}(P - P_{\mathcal{S}_t\infty})$ does not depend on $t \in [\alpha, \infty)$ as well, and we may replace the point β in (3.10) by α. This yields formula (3.2). Consequently, by using $\operatorname{rank} \mathcal{G}^\infty = \operatorname{rank} P$ and (2.34) we have

$$\begin{aligned} \operatorname{rank} \hat{N}_\infty &= \operatorname{rank} T_{\alpha,\infty} + (\operatorname{rank} P - \operatorname{rank} P_{\mathcal{S}_\alpha\infty}) \\ &= \operatorname{rank} T_{\alpha,\infty} + \operatorname{rank} \mathcal{G}^\infty - \big(n - d[\alpha, \infty)\big). \end{aligned} \tag{3.11}$$

This shows the validity of (3.3). Finally, by using (3.11) and (3.1) we have

$$\operatorname{rank} \mathcal{G}^\infty - \operatorname{rank} \hat{N}_\infty \overset{(3.11)}{=} n - d[\alpha, \infty) - \operatorname{rank} T_{\alpha,\infty} \overset{(3.1)}{=} m_L(\infty),$$

showing formula (3.4). The proof is complete. □

We comment the results in Theorem 3.3 in the following remarks.

Remark 3.4. We note that formulas (3.2) and (3.4) do not depend on the particular choice of the minimal principal solution $\hat{Y}_\infty$. More precisely, if $\hat{Y}_{\infty *}$ is another minimal principal solution of (H) at ∞, then according to Proposition 2.9 there exists a constant nonsingular matrix $K \in \mathbb{R}^{n\times n}$ such that $\hat{Y}_{\infty *}(t) = \hat{Y}_\infty(t)\,K$ for all $t \in \mathcal{I}$, and hence $W(\hat{Y}_{\infty *}, Y) = K^T\,W(\hat{Y}_\infty, Y)$. Thus, we have $\operatorname{Im}[W(\hat{Y}_{\infty *}, Y)]^T = \operatorname{Im}[W(\hat{Y}_\infty, Y)]^T$ and $\operatorname{rank} W(\hat{Y}_{\infty *}, Y) = \operatorname{rank} W(\hat{Y}_\infty, Y)$.

Remark 3.5. The multiplicity of the left proper focal point of Y at $t_0 \in \mathcal{I}$ satisfies

$$m_L(t_0) \overset{(1.2)}{=} \operatorname{rank} X(t_0^-) - \operatorname{rank} X(t_0) = \operatorname{rank} \mathcal{G}(t_0^-) - \operatorname{rank} W(\hat{Y}_{t_0}, Y), \tag{3.12}$$

where $\mathcal{G}(t_0^-)$ is the genus of Y in the left neighborhood of t_0 according to Remark 2.6, and where $W(\hat{Y}_{t_0}, Y) = -X(t_0)$. Formula (3.4) is then a counterpart of (3.12) for the case of $t_0 = \infty$. In this respect the expressions in (3.4) and (3.12) represent natural interpretations of the definitions of the multiplicity of the (left) proper focal point of Y at $t_0 \in \mathcal{I} \cup \{\infty\}$, which do not use the actual value of $X(t)$ at t_0.

Remark 3.6. Similarly as in (3.1) we define for $\mathcal{I} = (-\infty, b]$ the multiplicity of the (right) proper focal point of Y at $-\infty$ by

$$m_R(-\infty) := n - d(-\infty, \beta] - \operatorname{rank} T_{\beta,\,-\infty}, \tag{3.13}$$

where $\beta \in (-\infty, b]$ is such that the conjoined basis Y has constant kernel on $(-\infty, \beta]$ and where the matrix $T_{\beta,\,-\infty}$ is defined in (2.44). The definition in (3.13) is correct under assumption (1.1), as we also comment in Remark 3.2(i). Then $0 \le m_R(-\infty) \le n - d_{-\infty}$ with $d_{-\infty}$ defined in (2.30). In this case $m_R(-\infty) = 0$ if and only if Y is an antiprincipal solution of (H) at $-\infty$, while $m_R(-\infty) = n - d_{-\infty}$ if and only if Y is a principal solution of (H) at $-\infty$. Finally, if Y belongs to a genus $\mathcal{G}^{-\infty}$ near $-\infty$, then analogously to Theorem 3.3 we have the formula

$$m_R(-\infty) = \operatorname{rank} \mathcal{G}^{-\infty} - \operatorname{rank} W(\hat{Y}_{-\infty}, Y), \tag{3.14}$$

where $\hat{Y}_{-\infty}$ is the minimal principal solution of (H) at $-\infty$.

In the final remark of this section we comment the above results for the case of a completely controllable system (H).

Remark 3.7. The idea to consider the multiplicity of a focal point of Y at ∞ is new in the theory of nonoscillatory linear Hamiltonian systems. It is new even for the completely controllable system (H) on $[a, \infty)$. In this case $d[\alpha, \infty) = 0 = d_\infty$ for all $\alpha \in [a, \infty)$ and there is only one genus $\mathcal{G}^\infty = \mathcal{G}^\infty_{\min} = \mathcal{G}^\infty_{\max}$ of conjoined bases of (H), which satisfies $\operatorname{rank} \mathcal{G}^\infty = n$. Let Y be a conjoined basis of (H) with $X(t)$ invertible on an interval $[\alpha, \infty)$ and let $T_{\alpha,\,\infty}$ be the corresponding matrix in (1.7). According to (3.1) we define in this case

$$m_L(\infty) := n - \operatorname{rank} T_{\alpha,\,\infty} = \operatorname{def} T_{\alpha,\,\infty}. \tag{3.15}$$

The results in (3.2) and (3.4) in Theorem 3.3 then reduce to the equalities

$$\operatorname{Im} [W(\hat{Y}_\infty, Y)]^T = \operatorname{Im} T_{\alpha,\,\infty}, \quad \operatorname{rank} T_{\alpha,\,\infty} = \operatorname{rank} W(\hat{Y}_\infty, Y), \tag{3.16}$$

$$m_L(\infty) = n - \operatorname{rank} W(\hat{Y}_\infty, Y) = \operatorname{def} W(\hat{Y}_\infty, Y) \tag{3.17}$$

In particular, in this case the number $m_L(\infty)$ measures “how much” the conjoined basis Y and the principal solution $\hat{Y}_\infty$ at ∞ are linearly dependent. In a similar way, if (H) is completely controllable on the interval $(-\infty, b]$, then $d(-\infty, \beta] = 0 = d_{-\infty}$ for all $\beta \in (-\infty, b]$. And if Y is a conjoined basis of (H) with $X(t)$ invertible on $(-\infty, \beta]$, then by (3.13) we define

$$m_R(-\infty) := n - \operatorname{rank} T_{\beta,\,-\infty} = \operatorname{def} T_{\beta,\,-\infty}. \tag{3.18}$$

Analogously to (3.16) and (3.17) we then have

$$\operatorname{Im}[W(\hat{Y}_{-\infty}, Y)]^T = \operatorname{Im} T_{\beta,\,-\infty}, \quad \operatorname{rank} T_{\beta,\,-\infty} = \operatorname{rank} W(\hat{Y}_{-\infty}, Y),$$
$$m_R(-\infty) = n - \operatorname{rank} W(\hat{Y}_{-\infty}, Y) = \operatorname{def} W(\hat{Y}_{-\infty}, Y).$$

We also note that the considerations in this remark are new also for the even order Sturm–Liouville differential equations, being a special case of a completely controllable linear Hamiltonian system (H), see Section 7.

4. Comparative index on unbounded interval

In this section we analyze the properties of the comparative index for two conjoined bases of (H) on an unbounded interval. In particular, we study in detail the situation when one of the conjoined bases is $\hat{Y}_\infty$, i.e., the minimal principal solution of (H) at ∞. The results in this section extend the results from [30, Section 4] to an unbounded interval. First we derive a simple statement about Wronskians involving $\hat{Y}_\infty$. It can be regarded as an extension of [30, Lemma 3.2] to the case of $b = \infty$.

Lemma 4.1. *Assume that (1.1) holds with $\mathcal{I} = [a, \infty)$ and system (H) is nonoscillatory at ∞. Let Y be a conjoined basis of (H) with constant kernel on $[\alpha, \infty) \subseteq \mathcal{I}$ and let $\bar{Y}$ be a conjoined basis of (H) satisfying (2.5) and (2.35). Then the matrix $X(t)\,\bar{X}^T(t) \geq 0$ for every $t \in [\alpha, \infty)$. Moreover, if $\hat{Y}_\infty$ is the minimal principal solution of (H) at ∞, then*

$$W(\hat{Y}_\infty, Y)\,[W(\hat{Y}_\infty, \bar{Y})]^T \geq 0. \tag{4.1}$$

Proof. By [30, Lemma 3.2] we know that $X(t)\,\bar{X}^T(t) \geq 0$ on $[\alpha, b]$ for every $b \in (\alpha, \infty)$, which yields the first part of the theorem. From (2.6) with $\tilde{Y} := \hat{Y}_\infty$ we know that the matrix $W(\hat{Y}_\infty, Y)\,[W(\hat{Y}_\infty, \bar{Y})]^T$ is symmetric. Next, choose $\beta \in [\alpha, \infty)$ such that $d[\beta, \infty) = d_\infty$ and the conjoined bases Y, $\bar{Y}$, and $\hat{Y}_\infty$ have constant kernel on $[\beta, \infty)$. Let $\bar{Y}_\beta$ be a conjoined basis of (H) satisfying (2.5) and (2.35) with $Y := \hat{Y}_\infty$ and $\alpha := \beta$. By Proposition 2.11 we know that $\bar{Y}_\beta$ is a maximal antiprincipal solution of (H) at ∞, while from formula (3.8) in the proof of Theorem 3.3 it follows that

$$W(\hat{Y}_\infty, Y) = \lim_{t\to\infty} \bar{X}_\beta^{-1}(t)\,X(t), \quad W(\hat{Y}_\infty, \bar{Y}) = \lim_{t\to\infty} \bar{X}_\beta^{-1}(t)\,\bar{X}(t). \tag{4.2}$$

Consequently, with the aid of (4.2) we obtain

$$W(\hat{Y}_\infty, Y)\,[W(\hat{Y}_\infty, \bar{Y})]^T = \lim_{t\to\infty} \bar{X}_\beta^{-1}(t)\,[X(t)\,\bar{X}^T(t)]\,\bar{X}_\beta^{T-1}(t) \geq 0.$$

This shows that (4.1) holds. $\square$

In the sequel we will use a symplectic fundamental matrix of nonoscillatory system (H) at ∞, which is determined by the minimal principal solution $\hat{Y}_\infty$. According to (2.4) we denote

$$\hat{\Phi}_\infty(t) := \begin{pmatrix} \hat{Y}_\infty(t) & \bar{Y}_\infty(t) \end{pmatrix}, \quad t \in [a, \infty), \tag{4.3}$$

where $\bar{Y}_\infty$ is a conjoined basis of (H) which is normalized with $\hat{Y}_\infty$, i.e., $W(\hat{Y}_\infty, \bar{Y}_\infty) = I$. Every conjoined basis Y of (H) can be then uniquely represented in the spirit of Proposition 2.5 via the fundamental matrix $\hat{\Phi}_\infty(t)$ and a constant $2n \times n$ matrix D_∞, that is,

$$Y(t) = \hat{\Phi}_\infty(t)\, D_\infty, \quad t \in [a, \infty), \quad D_\infty = \begin{pmatrix} -W(\bar{Y}_\infty, Y) \\ W(\hat{Y}_\infty, Y) \end{pmatrix}, \quad \mathcal{J} D_\infty = \begin{pmatrix} W(\hat{Y}_\infty, Y) \\ W(\bar{Y}_\infty, Y) \end{pmatrix}, \tag{4.4}$$

where the matrix $\mathcal{J}$ is given in (2.4) and $D_\infty = \hat{\Phi}_\infty^{-1}(t)\, Y(t) = -\mathcal{J}\, \hat{\Phi}_\infty^T(t)\, \mathcal{J} Y(t)$, see (2.38). The following result is a fundamental tool for the proof of the Sturmian separation theorem (Theorem 5.1) in the next section. It is formulated in terms of the matrix $\mathcal{J} D_\infty$ from (4.4).

Theorem 4.2. *Assume that (1.1) holds with $\mathcal{I} = [a, \infty)$ and system (H) is nonoscillatory at ∞. Let $\hat{Y}_\infty$ be the minimal principal solution of (H) at ∞ with the associated matrix $\hat{\Phi}_\infty(t)$ in (4.3). Moreover, let Y be a conjoined basis of (H) with constant kernel on $[\alpha, \infty) \subseteq \mathcal{I}$ belonging to a genus $\mathcal{G}^\infty$ and let $\hat{Y}_\alpha$ be the principal solution of (H) at the point α. Then*

$$\mu\big(\mathcal{J} D_\infty, \mathcal{J} D_\infty^\alpha\big) = \operatorname{rank} \mathcal{G}^\infty - \operatorname{rank} W(\hat{Y}_\infty, Y), \quad \mu^*\big(Y(\alpha), \hat{Y}_\infty(\alpha)\big) = 0, \tag{4.5}$$

where D_∞ and D_∞^α are the constant matrices in (4.4) corresponding to Y and $\hat{Y}_\alpha$.

Proof. Following (4.4), let the matrices D_∞ and D_∞^α be split into the $n \times n$ blocks as

$$D_\infty = \begin{pmatrix} M_\infty \\ N_\infty \end{pmatrix} = \begin{pmatrix} -W(\bar{Y}_\infty, Y) \\ W(\hat{Y}_\infty, Y) \end{pmatrix}, \quad D_\infty^\alpha = \begin{pmatrix} M_\infty^\alpha \\ N_\infty^\alpha \end{pmatrix} = \begin{pmatrix} -W(\bar{Y}_\infty, \hat{Y}_\alpha) \\ W(\hat{Y}_\infty, \hat{Y}_\alpha) \end{pmatrix}. \tag{4.6}$$

By (4.4) with D_∞ and D_∞^α in (4.6) and by the definition of the comparative index in (2.21)–(2.22) with $Y := \mathcal{J} D_\infty$ and $\tilde{Y} := \mathcal{J} D_\infty^\alpha$ (note that (2.20) holds) we get

$$\mu\big(\mathcal{J} D_\infty, \mathcal{J} D_\infty^\alpha\big) = \operatorname{rank} \mathcal{M} + \operatorname{ind} \mathcal{P}, \tag{4.7}$$

$$\mathcal{M} = (I - N_\infty^\dagger N_\infty)\, W, \quad \mathcal{P} = V\, W^T N_\infty^\dagger N_\infty^\alpha\, V, \quad V = I - \mathcal{M}^\dagger \mathcal{M}, \tag{4.8}$$

where $W := W\big(\mathcal{J} D_\infty, \mathcal{J} D_\infty^\alpha\big)$. We will show that $W = W(Y, \hat{Y}_\alpha)$. According to (2.3)–(2.4) and (4.4) we obtain for $t \in \mathcal{I}$ that

$$\begin{aligned} W(Y, \hat{Y}_\alpha) &\overset{(2.3)}{=} Y^T(t)\, \mathcal{J}\, \hat{Y}_\alpha(t) = D_\infty^T\, [\hat{\Phi}_\infty^T(t)\, \mathcal{J}\, \hat{\Phi}_\infty(t)]\, D_\infty^\alpha \overset{(2.4)}{=} D_\infty^T\, \mathcal{J}\, D_\infty^\alpha \\ &= D_\infty^T\, (\mathcal{J}^T \mathcal{J} \mathcal{J})\, D_\infty^\alpha = (\mathcal{J} D_\infty)^T \mathcal{J} (\mathcal{J} D_\infty^\alpha) \overset{(2.3)}{=} W\big(\mathcal{J} D_\infty, \mathcal{J} D_\infty^\alpha\big) = W. \end{aligned}$$

In particular, evaluating the Wronskian $W(Y, \hat{Y}_\alpha)$ at $t = \alpha$ and using (1.4) with $s := \alpha$ yields the equality $W = X^T(\alpha)$. Consequently, the first and the second identity in (4.8) then read as

$$\mathcal{M} = (I - N_\infty^\dagger N_\infty)\, X^T(\alpha), \quad \mathcal{P} = V X(\alpha)\, N_\infty^\dagger N_\infty^\alpha\, V. \tag{4.9}$$

Let P, $P_{\mathcal{S}_\alpha \infty}$, and $T_{\alpha,\infty}$ be the matrices in (2.31), (2.33), and (1.7), which correspond to Y. In accordance with Theorem 3.3 we have $\operatorname{Im} N_\infty^T = \operatorname{Im} T_{\alpha,\infty} \oplus \operatorname{Im}(P - P_{\mathcal{S}_\alpha \infty})$, while (2.34) implies $\operatorname{Im} T_{\alpha,\infty} \subseteq \operatorname{Im} P$. This yields that

$$\operatorname{Im} N_\infty^T \subseteq \operatorname{Im} P, \quad \text{i.e.,} \quad P N_\infty^\dagger N_\infty = N_\infty^\dagger N_\infty = N_\infty^\dagger N_\infty P. \tag{4.10}$$

And since $P = [X^\dagger(\alpha)\, X(\alpha)]^T$ by (2.31), the first equality in (4.9) yields that

$$\operatorname{Im} \mathcal{M} = \operatorname{Im}(I - N_\infty^\dagger N_\infty)\, P = \operatorname{Im}(P - N_\infty^\dagger N_\infty), \tag{4.11}$$

$$\operatorname{rank} \mathcal{M} = \operatorname{rank} P - \operatorname{rank} N_\infty = \operatorname{rank} \mathcal{G}^\infty - \operatorname{rank} N_\infty. \tag{4.12}$$

Next we will prove that the matrix V satisfies

$$\operatorname{Im} X^T(\alpha)\, V = \operatorname{Im} N_\infty^T, \quad \text{i.e.,} \quad X^T(\alpha)\, V = N_\infty^T\, K \tag{4.13}$$

for some invertible matrix K. Since $\mathcal{M}V = 0$, it follows that $N_\infty^\dagger N_\infty\, X^T(\alpha)\, V = X^T(\alpha)\, V$ by (4.9) and hence, $\operatorname{Im} X^T(\alpha)\, V \subseteq \operatorname{Im} N_\infty^\dagger N_\infty = \operatorname{Im} N_\infty^T$. Conversely, assume that $v \in \operatorname{Im} N_\infty^T$. Then we also have $v \in \operatorname{Im} P = \operatorname{Im} X^T(\alpha)$ and there exists $w \in \mathbb{R}^n$ such that $v = X^T(\alpha)\, w$. Then we write $X^T(\alpha)\, V w = X^T(\alpha)\, w - X^T(\alpha)\, \mathcal{M}^\dagger \mathcal{M} w = v - X^T(\alpha)\, \mathcal{M}^\dagger \mathcal{M} w$. But by using (4.9) we have $\mathcal{M} w = (I - N_\infty^\dagger N_\infty)\, X^T(\alpha)\, w = (I - N_\infty^\dagger N_\infty)\, v = 0$, so that $v = X^T(\alpha)\, V w$. Therefore, $v \in \operatorname{Im} X^T(\alpha)\, V$ and (4.13) is proven.

Let $\bar{Y}$ be a conjoined basis of (H) satisfying (2.5) and (2.35) and set $\bar{N}_\infty := W(\hat{Y}_\infty, \bar{Y})$. Since $W(\bar{Y}, \bar{Y}) = 0 = W(Y, Y)$ and $W(Y, \bar{Y}) = I$, according to formula (2.8) in Proposition 2.1 with $Y_1 := \hat{Y}_\infty$, $Y_2 := \bar{Y}$, $Y_3 := Y$, and $Y_4 := \hat{Y}_\alpha$ we obtain

$$W(\hat{Y}_\infty, \bar{Y})\, W(Y, \hat{Y}_\alpha) - W(\hat{Y}_\infty, Y)\, W(\bar{Y}, \hat{Y}_\alpha) = W(\hat{Y}_\infty, \hat{Y}_\alpha). \tag{4.14}$$

Following the above notation and using the facts that $W(Y, \hat{Y}_\alpha) = X^T(\alpha)$, $W(\bar{Y}, \hat{Y}_\alpha) = \bar{X}^T(\alpha)$, and $N_\infty^\alpha = W(\hat{Y}_\infty, \hat{Y}_\alpha) = \hat{X}_\infty^T(\alpha)$, identity (4.14) then has the form

$$\bar{N}_\infty\, X^T(\alpha) - N_\infty\, \bar{X}^T(\alpha) = N_\infty^\alpha = \hat{X}_\infty^T(\alpha). \tag{4.15}$$

Combining the formula for $\mathcal{P}$ in (4.9) with equalities (4.13) and (4.15) and with the identities $X(\alpha) = X(\alpha)\, P$, $\bar{N}_\infty N_\infty^T = N_\infty \bar{N}_\infty^T$, and $P \bar{X}^T(\alpha) = 0$ then implies that

$$\begin{aligned}
\mathcal{P} &\overset{(4.9)}{=} V X(\alpha)\, N_\infty^\dagger N_\infty^\alpha\, V \overset{(4.15)}{=} V X(\alpha)\, N_\infty^\dagger\, [\bar{N}_\infty\, X^T(\alpha) - N_\infty\, \bar{X}^T(\alpha)]\, V \\
&\overset{(4.13),\,(4.10)}{=} K^T N_\infty N_\infty^\dagger \bar{N}_\infty N_\infty^T\, K - V X(\alpha)\, N_\infty^\dagger N_\infty P \bar{X}^T(\alpha)\, V \\
&= K^T N_\infty N_\infty^\dagger N_\infty \bar{N}_\infty^T\, K = K^T N_\infty \bar{N}_\infty^T\, K \geq 0,
\end{aligned}$$

where the last inequality follows from (4.1) in Lemma 4.1. Therefore, $\operatorname{ind}\mathcal{P} = 0$. Upon combining (4.7) and (4.12) we get $\mu\big(\mathcal{J}D_\infty, \mathcal{J}D_\infty^\alpha\big) = \operatorname{rank}\mathcal{M} = \operatorname{rank}\mathcal{G}^\infty - \operatorname{rank} N_\infty$. This shows the first formula in (4.5). For the second formula in (4.5) we have by (2.21)–(2.22) with $Y := Y(\alpha)$ and $\tilde{Y} := \hat{Y}_\infty(\alpha)$

$$\mu^*\big(Y(\alpha), \hat{Y}_\infty(\alpha)\big) = \operatorname{rank}\mathcal{M} + \operatorname{ind}(-\mathcal{P}), \tag{4.16}$$

$$\mathcal{M} = (I - P)\, W, \quad \mathcal{P} = V\, W^T X^\dagger(\alpha)\, \hat{X}_\infty(\alpha)\, V, \quad V = I - \mathcal{M}^\dagger \mathcal{M}, \tag{4.17}$$

where $W = W(Y, \hat{Y}_\infty) = -N_\infty^T$ by (4.6). According to (4.10) we have $\mathcal{M} = 0$, so that $V = I$ and $\mathcal{P} = -N_\infty\, X^\dagger(\alpha)\, \hat{X}_\infty(\alpha)$ by (4.17). Consequently, using the value $\hat{X}_\infty(\alpha)$ in (4.15) we get

$$\begin{aligned} -\mathcal{P} &= N_\infty\, X^\dagger(\alpha)\, [\bar{N}_\infty\, X^T(\alpha) - N_\infty\, \bar{X}^T(\alpha)]^T = N_\infty P\, \bar{N}_\infty^T - N_\infty\, X^\dagger(\alpha)\, \bar{X}(\alpha)\, N_\infty^T \\ &\overset{(2.35)}{=} N_\infty P\, \bar{N}_\infty^T \overset{(4.10)}{=} N_\infty \bar{N}_\infty^T \geq 0, \end{aligned}$$

where the last inequality follows from Lemma 4.1. Therefore, $\operatorname{ind}(-\mathcal{P}) = 0$ and hence, equation (4.16) yields that $\mu^*\big(Y(\alpha), \hat{Y}_\infty(\alpha)\big) = 0$. The proof is complete. □

Remark 4.3. The second formula in (4.5) follows also from the algebraic properties of the comparative index in [10, Property 3, p. 448] and from Lemma 4.1. The first property in (4.5) can be then obtained by (2.24) and [10, Property 3, p. 448] again as follows:

$$\begin{aligned} \mu\big(\mathcal{J}D_\infty, \mathcal{J}D_\infty^\alpha\big) &\overset{(2.24)}{=} \operatorname{rank} X(\alpha) - \mu\big(\mathcal{J}D_\infty^\alpha, \mathcal{J}D_\infty\big) \\ &= \operatorname{rank}\mathcal{G}^\infty - \mu\big(\mathcal{J}\hat{\Phi}_\infty^{-1}(\alpha)\, E, \mathcal{J}\hat{\Phi}_\infty^{-1}(\alpha)\, \Phi(\alpha)\, \mathcal{J}E\big) \\ &\overset{[10]}{=} \operatorname{rank}\mathcal{G}^\infty - \mu^*\big(\hat{\Phi}_\infty(\alpha)\, \mathcal{J}E, \Phi(\alpha)\, \mathcal{J}E\big) = \operatorname{rank}\mathcal{G}^\infty - \mu^*\big(\hat{Y}_\infty(\alpha), Y(\alpha)\big) \\ &\overset{(2.24)}{=} \operatorname{rank}\mathcal{G}^\infty - \operatorname{rank} W(Y(\alpha), \hat{Y}_\infty(\alpha)) + \mu^*\big(Y(\alpha), \hat{Y}_\infty(\alpha)\big), \end{aligned} \tag{4.18}$$

where the symplectic fundamental matrices $\hat{\Phi}_\infty(t)$ and $\Phi(t)$ are given in (4.3) and (2.4). Now the last term in (4.18) is zero, while the middle term is equal to the rank of $W(\hat{Y}_\infty, Y)$. Therefore, we obtain the first property in (4.5) from (4.18).

In view of Theorem 3.3 we can now interpret the result of Theorem 4.2 as an analogue of [30, Corollary 4.5] with $b := \infty$.

Corollary 4.4. *With the assumptions and notation in Theorem 4.2 we have*

$$m_L(\alpha, \infty] = \mu\big(\mathcal{J}D_\infty, \mathcal{J}D_\infty^\alpha\big), \quad m_R[\alpha, \infty) = \mu^*\big(Y(\alpha), \hat{Y}_\infty(\alpha)\big). \tag{4.19}$$

Proof. Since the conjoined basis Y has constant kernel on $[\alpha, \infty)$, we have $m_L(\alpha, \infty) = 0$ and $m_R[\alpha, \infty) = 0$. The formulas in (4.19) now follow from (3.4) and (4.5). □

In the last result of this section we will consider two conjoined bases Y and $\tilde{Y}$ of (H) with constant kernel on $[\alpha, \infty)$. Namely, we calculate the difference of their comparative indices in the spirit of Corollary 4.4.

Lemma 4.5. *Assume that (1.1) holds with $\mathcal{I} = [a, \infty)$ and system (H) is nonoscillatory at ∞. Let Y and $\tilde{Y}$ be conjoined bases of (H) with constant kernel on $[\alpha, \infty) \subseteq \mathcal{I}$. Then*

$$m_L(\alpha, \infty] - \widetilde{m}_L(\alpha, \infty] = \mu\big(\mathcal{J}D_\infty, \mathcal{J}\tilde{D}_\infty\big) - \mu\big(Y(\alpha), \tilde{Y}(\alpha)\big), \tag{4.20}$$

$$m_R[\alpha, \infty) - \widetilde{m}_R[\alpha, \infty) = \mu^*\big(Y(\alpha), \tilde{Y}(\alpha)\big) - \mu^*\big(\mathcal{J}D_\infty, \mathcal{J}\tilde{D}_\infty\big), \tag{4.21}$$

where D_∞ and $\tilde{D}_\infty$ are the constant matrices in (4.4) corresponding to Y and $\tilde{Y}$.

Proof. We define the constant symplectic matrix $\mathcal{S} := -\mathcal{J}\,\hat{\Phi}_\infty^{-1}(\alpha)$, where $\hat{\Phi}_\infty(t)$ is given in (4.3), and consider the symplectic fundamental matrices $\Phi(t)$ and $\tilde{\Phi}(t)$ of (H) such that $\Phi(t)\,E = Y(t)$ and $\tilde{\Phi}(t)\,E = \tilde{Y}(t)$ on $[a, \infty)$, with the constant $2n \times n$ matrix E from (2.25). In addition, let D_∞^α be the matrix in (4.4) corresponding to the principal solution $\hat{Y}_\alpha$ of (H) at the point α. Then we have $Y(t) = \hat{\Phi}_\infty(t)\,D_\infty$, $\tilde{Y}(t) = \hat{\Phi}_\infty(t)\,\tilde{D}_\infty$, and $\hat{Y}_\alpha(t) = \hat{\Phi}_\infty(t)\,D_\infty^\alpha$ on $[a, \infty)$. It follows that for $t = \alpha$ we have

$$\mathcal{J}D_\infty = -\mathcal{S}\,\Phi(\alpha)\,E, \quad \mathcal{J}\tilde{D}_\infty = -\mathcal{S}\,\tilde{\Phi}(\alpha)\,E, \quad \mathcal{J}D_\infty^\alpha = -\mathcal{S}E. \tag{4.22}$$

By Corollary 4.4 applied to Y and $\tilde{Y}$ together with transformation formula (2.26) with the matrices $\mathcal{S} := -\mathcal{S}$, $\Phi := \Phi(\alpha)$, and $\tilde{\Phi} := \tilde{\Phi}(\alpha)$ we obtain

$$\begin{aligned}
m_L(\alpha, \infty] - \widetilde{m}_L(\alpha, \infty] &\overset{(4.19)}{=} \mu\big(\mathcal{J}D_\infty, \mathcal{J}D_\infty^\alpha\big) - \mu\big(\mathcal{J}\tilde{D}_\infty, \mathcal{J}D_\infty^\alpha\big) \\
&\overset{(4.22)}{=} \mu\big(-\mathcal{S}\,\Phi(\alpha)\,E, -\mathcal{S}E\big) - \mu\big(-\mathcal{S}\,\tilde{\Phi}(\alpha)\,E, -\mathcal{S}E\big) \\
&\overset{(2.26)}{=} \mu\big(-\mathcal{S}\,\Phi(\alpha)\,E, -\mathcal{S}\,\tilde{\Phi}(\alpha)\,E\big) - \mu\big(\Phi(\alpha)\,E, \tilde{\Phi}(\alpha)\,E\big) \\
&\overset{(4.22)}{=} \mu\big(\mathcal{J}D_\infty, \mathcal{J}\tilde{D}_\infty\big) - \mu\big(Y(\alpha), \tilde{Y}(\alpha)\big),
\end{aligned}$$

which shows formula (4.20). Next, by (4.3) we know that $\hat{\Phi}_\infty(t)\,\mathcal{J}E = \hat{Y}_\infty(t)$ on $[a, \infty)$. Thus,

$$\hat{Y}_\infty(\alpha) = \mathcal{S}^{-1}E, \quad Y(\alpha) = \mathcal{S}^{-1}\mathcal{S}\,\Phi(\alpha)\,E, \quad \tilde{Y}(\alpha) = \mathcal{S}^{-1}\mathcal{S}\,\tilde{\Phi}(\alpha)\,E. \tag{4.23}$$

Combining Corollary 4.4 applied to Y and $\tilde{Y}$ and transformation formula (2.27) with the matrices $\mathcal{S} := \mathcal{S}^{-1}$, $\Phi := \mathcal{S}\,\Phi(\alpha)$, and $\tilde{\Phi} := \mathcal{S}\,\tilde{\Phi}(\alpha)$ then yields

$$\begin{aligned}
m_R[\alpha, \infty) - \widetilde{m}_R[\alpha, \infty) &\overset{(4.19)}{=} \mu^*\big(Y(\alpha), \hat{Y}_\infty(\alpha)\big) - \mu^*\big(\tilde{Y}(\alpha), \hat{Y}_\infty(\alpha)\big) \\
&\overset{(4.23)}{=} \mu^*\big(\mathcal{S}^{-1}\mathcal{S}\,\Phi(\alpha)\,E, \mathcal{S}^{-1}E\big) - \mu^*\big(\mathcal{S}^{-1}\mathcal{S}\,\tilde{\Phi}(\alpha)\,E, \mathcal{S}^{-1}E\big) \\
&\overset{(2.27)}{=} \mu^*\big(\mathcal{S}^{-1}\mathcal{S}\,\Phi(\alpha)\,E, \mathcal{S}^{-1}\mathcal{S}\,\tilde{\Phi}(\alpha)\,E\big) - \mu^*\big(\mathcal{S}\,\Phi(\alpha)\,E, \mathcal{S}\,\tilde{\Phi}(\alpha)\,E\big) \\
&\overset{(4.23),\,(4.22)}{=} \mu^*\big(Y(\alpha), \tilde{Y}(\alpha)\big) - \mu^*\big(-\mathcal{J}D_\infty, -\mathcal{J}\tilde{D}_\infty\big) \\
&\overset{(2.28)}{=} \mu^*\big(Y(\alpha), \tilde{Y}(\alpha)\big) - \mu^*\big(\mathcal{J}D_\infty, \mathcal{J}\tilde{D}_\infty\big),
\end{aligned}$$

which shows formula (4.21). The proof is complete. □

5. Singular Sturmian separation theorems

In this section we derive Sturmian separation theorems for the numbers of left proper focal points, resp. right proper focal points, of two conjoined bases of a nonoscillatory system (H) on the unbounded intervals $(a,\infty]$ or $(-\infty,b]$, resp. $[a,\infty)$ or $[-\infty,b)$. The results regarding left proper focal points include the multiplicities of focal points at ∞ as we discussed in Section 3, while the results regarding right proper focal points include the multiplicities of focal points at $-\infty$. In addition, we will also derive the corresponding results for the open intervals (a,∞) or $(-\infty,b)$. As in the previous sections we do not impose any controllability assumption and the results are new even for a completely controllable system (H).

The following result corresponds to formulas (2.15) and (2.16) in the case of a compact interval $\mathcal{I}=[a,b]$, see [30, Theorem 4.1] and Remark 5.2 below.

Theorem 5.1 *(Singular Sturmian separation theorem). Assume that* (1.1) *holds with $\mathcal{I}=[a,\infty)$ and system* (H) *is nonoscillatory at ∞. Then for any conjoined bases Y and $\tilde{Y}$ of* (H) *we have the equalities*

$$m_L(a,\infty]-\widetilde{m}_L(a,\infty]=\mu\big(\mathcal{J}D_\infty,\mathcal{J}\tilde{D}_\infty\big)-\mu\big(Y(a),\tilde{Y}(a)\big), \tag{5.1}$$

$$m_R[a,\infty)-\widetilde{m}_R[a,\infty)=\mu^*\big(Y(a),\tilde{Y}(a)\big)-\mu^*\big(\mathcal{J}D_\infty,\mathcal{J}\tilde{D}_\infty\big), \tag{5.2}$$

where D_∞ and $\tilde{D}_\infty$ are the constant matrices in (4.4) *corresponding to Y and $\tilde{Y}$.*

Proof. Let $\alpha\in\mathcal{I}$ be such that both conjoined bases Y and $\tilde{Y}$ have constant kernel on the interval $[\alpha,\infty)$. Applying formulas (2.15)–(2.16) on the interval $[a,\alpha]$ we obtain that

$$m_L(a,\alpha]-\widetilde{m}_L(a,\alpha]=\mu\big(Y(\alpha),\tilde{Y}(\alpha)\big)-\mu\big(Y(a),\tilde{Y}(a)\big), \tag{5.3}$$

$$m_R[a,\alpha)-\widetilde{m}_R[a,\alpha)=\mu^*\big(Y(a),\tilde{Y}(a)\big)-\mu^*\big(Y(\alpha),\tilde{Y}(\alpha)\big). \tag{5.4}$$

On the other hand, according to (4.20)–(4.21) in Lemma 4.5 we have the identities

$$m_L(\alpha,\infty]-\widetilde{m}_L(\alpha,\infty]=\mu\big(\mathcal{J}D_\infty,\mathcal{J}\tilde{D}_\infty\big)-\mu\big(Y(\alpha),\tilde{Y}(\alpha)\big), \tag{5.5}$$

$$m_R[\alpha,\infty)-\widetilde{m}_R[\alpha,\infty)=\mu^*\big(Y(\alpha),\tilde{Y}(\alpha)\big)-\mu^*\big(\mathcal{J}D_\infty,\mathcal{J}\tilde{D}_\infty\big). \tag{5.6}$$

Since $m_L(a,\infty]=m_L(a,\alpha]+m_L(\alpha,\infty]$ and $\widetilde{m}_L(a,\infty]=\widetilde{m}_L(a,\alpha]+\widetilde{m}_L(\alpha,\infty]$, adding equalities (5.3) and (5.5) yields the formula in (5.1). Similarly, from $m_R[a,\infty)=m_R[a,\alpha)+m_R[\alpha,\infty)$ and $\widetilde{m}_R[a,\infty)=\widetilde{m}_R[a,\alpha)+\widetilde{m}_R[\alpha,\infty)$ together with (5.4) and (5.6) we obtain the formula in (5.2). The proof is complete. □

Remark 5.2. The results in Theorem 5.1 represent the true “singular version” of the formulas in (2.15) and (2.16), which deal with a compact interval $\mathcal{I}=[a,b]$. Indeed, for a fixed point $s\in[a,b]$ we consider the (symplectic) fundamental matrix $\hat{\Phi}_s(t)$ of system (H) in the form

$$\hat{\Phi}_s(t):=\begin{pmatrix}\hat{Y}_s(t) & \bar{Y}_s(t)\end{pmatrix},\quad t\in[a,b], \tag{5.7}$$

where $\hat{Y}_s$ is the principal solution of (H) at the point s and where $\bar{Y}_s$ completes $\hat{Y}_s$ to a normalized pair of conjoined bases, i.e., $W(\hat{Y}_s, \bar{Y}_s) = I$. Here we take the conjoined basis $\bar{Y}_s$ with the initial conditions $\bar{Y}_s(s) = -\mathcal{J}E = (-I,\ 0)^T$, so that $\hat{\Phi}_s(s) = -\mathcal{J}$. Then for every conjoined basis Y of (H) there exists a unique constant $2n \times n$ representation matrix D_s such that, in spirit of Proposition 2.5,

$$Y(t) = \hat{\Phi}_s(t)\, D_s, \quad t \in [a,b], \quad D_s = \begin{pmatrix} -W(\bar{Y}_s, Y) \\ W(\hat{Y}_s, Y) \end{pmatrix}, \quad \mathcal{J}D_s = \begin{pmatrix} W(\hat{Y}_s, Y) \\ W(\bar{Y}_s, Y) \end{pmatrix}. \tag{5.8}$$

Using (5.7) and (5.8) with $s := a$ and $s := b$ yields that $Y(a) = -\mathcal{J}D_a$ and $Y(b) = -\mathcal{J}D_b$. Similarly, for another conjoined basis $\tilde{Y}$ of (H) we have $\tilde{Y}(a) = -\mathcal{J}\tilde{D}_a$ and $\tilde{Y}(b) = -\mathcal{J}\tilde{D}_b$. Therefore, by using property (2.28) of the comparative index, the formulas (2.15) and (2.16) can be rewritten as

$$m_L(a,b] - \widetilde{m}_L(a,b] = \mu(\mathcal{J}D_b, \mathcal{J}\tilde{D}_b) - \mu(\mathcal{J}D_a, \mathcal{J}\tilde{D}_a),$$
$$m_R[a,b) - \widetilde{m}_R[a,b) = \mu^*(\mathcal{J}D_a, \mathcal{J}\tilde{D}_a) - \mu^*(\mathcal{J}D_b, \mathcal{J}\tilde{D}_b),$$

while the formulas (5.1) and (5.2) in Theorem 5.1 can be rewritten as

$$m_L(a,\infty] - \widetilde{m}_L(a,\infty] = \mu(\mathcal{J}D_\infty, \mathcal{J}\tilde{D}_\infty) - \mu(\mathcal{J}D_a, \mathcal{J}\tilde{D}_a),$$
$$m_R[a,\infty) - \widetilde{m}_R[a,\infty) = \mu^*(\mathcal{J}D_a, \mathcal{J}\tilde{D}_a) - \mu^*(\mathcal{J}D_\infty, \mathcal{J}\tilde{D}_\infty).$$

By considering a special choice of the conjoined basis $\tilde{Y}$ in Theorem 5.1 we obtain the following formulas. They highlight the importance of the minimal principal solution $\hat{Y}_\infty$ of (H) at ∞ in counting the exact number of left and right proper focal points of any conjoined basis Y of (H) in the intervals $(a,\infty]$ and $[a,\infty)$. Also, they correspond to formulas (2.17) and (2.18) in the case of a compact interval $\mathcal{I} = [a,b]$, see [30, Equation (5.28)].

Corollary 5.3. *Assume that* (1.1) *holds with $\mathcal{I} = [a,\infty)$ and system* (H) *is nonoscillatory at ∞. Then for any conjoined basis Y of* (H) *we have the equalities*

$$m_L(a,\infty] = \widehat{m}_{La}(a,\infty] + \mu\big(\mathcal{J}D_\infty, \mathcal{J}D_\infty^a\big), \tag{5.9}$$
$$m_R[a,\infty) = \widehat{m}_{R\infty}[a,\infty) + \mu^*\big(Y(a), \hat{Y}_\infty(a)\big), \tag{5.10}$$

where D_∞ and D_∞^a are the constant matrices in (4.4) *corresponding to Y and $\hat{Y}_a$.*

Proof. Formula (5.9) follows from (5.1) with $\tilde{Y} := \hat{Y}_a$, since in this case $\tilde{D}_\infty = \hat{D}_\infty^a$ with the notation used in (4.4) and (4.6), and $\mu\big(Y(a), \hat{Y}_a(a)\big) = \mu\big(Y(a), E\big) = 0$ by (2.25). Similarly, formula (5.10) follows from (5.2) with $\tilde{Y} := \hat{Y}_\infty$, since in this case $\tilde{D}_\infty = (I,\ 0)^T$ and hence, $\mu^*(\mathcal{J}D_\infty, \mathcal{J}\tilde{D}_\infty) = \mu^*(\mathcal{J}D_\infty, -E) = 0$. □

In the next statement we connect the multiplicities of left and right proper focal points of one conjoined basis Y of (H) in an unbounded interval. This result corresponds to formula (2.11), see also [30, Theorem 5.1].

Theorem 5.4. *Assume that (1.1) holds with $\mathcal{I} = [a, \infty)$ and system (H) is nonoscillatory at ∞. Let $\hat{Y}_\infty$ be the minimal principal solution of (H) at ∞. Then for any conjoined basis Y of (H) its numbers of left proper focal points in the interval $(a, \infty]$ and right proper focal points in the interval $[a, \infty)$ satisfy*

$$m_L(a, \infty] + \operatorname{rank} W(\hat{Y}_\infty, Y) = m_R[a, \infty) + \operatorname{rank} X(a). \tag{5.11}$$

Proof. Let Y be a fixed conjoined basis of (H). According to Proposition 2.4 we choose $\alpha \geq a$ such that Y has constant kernel on $[\alpha, \infty)$. By (2.11) applied to the interval $[a, \alpha]$ we get

$$m_L(a, \alpha] + \operatorname{rank} X(\alpha) = m_R[a, \alpha) + \operatorname{rank} X(a). \tag{5.12}$$

In particular, the equalities $m_L(a, \alpha] = m_L(a, \infty)$, $m_R[a, \alpha) = m_R[a, \infty)$, and $\operatorname{rank} X(\alpha) = \operatorname{rank} \mathcal{G}^\infty$ hold, where $\mathcal{G}^\infty$ is the genus corresponding to Y. Then formula (5.12) reads as

$$m_L(a, \infty) + \operatorname{rank} \mathcal{G}^\infty = m_R[a, \infty) + \operatorname{rank} X(a). \tag{5.13}$$

Finally, combining formula (3.4) in Theorem 3.3 and equality (5.13) yields

$$\begin{aligned} m_L(a, \infty] + \operatorname{rank} W(\hat{Y}_\infty, Y) &= m_L(a, \infty) + m_L(\infty) + \operatorname{rank} W(\hat{Y}_\infty, Y) \\ &\overset{(3.4)}{=} m_L(a, \infty) + \operatorname{rank} \mathcal{G}^\infty \overset{(5.13)}{=} m_R[a, \infty) + \operatorname{rank} X(a), \end{aligned}$$

which shows identity (5.11). The proof is complete. □

In the remaining results of this section we will use the principal solution $\hat{Y}_a$ of (H) at the point a and the minimal principal solution $\hat{Y}_\infty$ of (H) at ∞. For these particular conjoined bases of (H) we have $W(\hat{Y}_\infty, \hat{Y}_a) = \hat{X}_\infty^T(a)$ and the statement of Theorem 5.4 yields

$$\widehat{m}_{L\infty}(a, \infty] = \widehat{m}_{R\infty}[a, \infty) + \operatorname{rank} \hat{X}_\infty(a), \tag{5.14}$$

$$\widehat{m}_{La}(a, \infty] + \operatorname{rank} \hat{X}_\infty(a) = \widehat{m}_{Ra}[a, \infty). \tag{5.15}$$

In the following statement we relate the numbers of left and right proper focal points of $\hat{Y}_\infty$ and $\hat{Y}_a$ in $(a, \infty]$ and $[a, \infty)$. This result corresponds to formulas (2.12) and (2.13), see also [30, Theorem 5.3, Corollary 5.4].

Theorem 5.5. *Assume that (1.1) holds with $\mathcal{I} = [a, \infty)$ and system (H) is nonoscillatory at ∞. Then we have*

$$\widehat{m}_{L\infty}(a, \infty] = \widehat{m}_{La}(a, \infty] + \operatorname{rank} \hat{X}_\infty(a), \quad \widehat{m}_{Ra}[a, \infty) = \widehat{m}_{R\infty}[a, \infty) + \operatorname{rank} \hat{X}_\infty(a), \tag{5.16}$$

$$\widehat{m}_{La}(a, \infty] = \widehat{m}_{R\infty}[a, \infty), \quad \widehat{m}_{Ra}[a, \infty) = \widehat{m}_{L\infty}(a, \infty]. \tag{5.17}$$

Proof. The first formula in (5.16) follows from (5.9) with $Y := \hat{Y}_\infty$. Indeed, in this case we have with the notation used in (4.4) and (4.6) that

$$\mathcal{J}D_\infty = -E, \quad \mathcal{J}D_\infty^a = \begin{pmatrix} W(\hat{Y}_\infty, \hat{Y}_a) \\ W(\bar{Y}_\infty, \hat{Y}_a) \end{pmatrix} = \begin{pmatrix} \hat{X}_\infty^T(a) \\ \bar{X}_\infty^T(a) \end{pmatrix}. \tag{5.18}$$

Hence, $\mu(\mathcal{J}D_\infty, \mathcal{J}D_\infty^a) = \mu(-E, \mathcal{J}D_\infty^a) = \operatorname{rank} \hat{X}_\infty(a)$ by (2.25). Similarly, the second formula in (5.16) follows from (5.10) with $Y := \hat{Y}_a$, since in this case $\mu^*\big(\hat{Y}_a(a), \hat{Y}_\infty(a)\big) = \mu^*\big(E, \hat{Y}_\infty(a)\big) = \operatorname{rank} \hat{X}_\infty(a)$ by (2.25). Finally, the formulas in (5.17) follow directly from (5.15) and (5.16). □

Remark 5.6. The results in Theorem 5.5 yield interesting connections with the limits of the corresponding equalities in (2.13). First of all, it is not at all clear whether the limits

$$\lim_{b\to\infty} \widehat{m}_{Lb}(a,b], \quad \lim_{b\to\infty} \widehat{m}_{Rb}[a,b) \tag{5.19}$$

exist, and if they exist, then what are their values. Below we show that both of these limits indeed exist and that the first one is equal to $\widehat{m}_{L\infty}(a,\infty]$ as we would formally expect, but surprisingly the second one is not equal to $\widehat{m}_{R\infty}[a,\infty)$ in general. More precisely, we have

$$\lim_{b\to\infty} \widehat{m}_{Lb}(a,b] \overset{(2.13)}{=} \lim_{b\to\infty} \widehat{m}_{Ra}[a,b) = \widehat{m}_{Ra}[a,\infty) \overset{(5.17)}{=} \widehat{m}_{L\infty}(a,\infty],$$

$$\lim_{b\to\infty} \widehat{m}_{Rb}[a,b) \overset{(2.13)}{=} \lim_{b\to\infty} \widehat{m}_{La}(a,b] = \widehat{m}_{La}(a,\infty) \overset{(5.17)}{=} \widehat{m}_{R\infty}[a,\infty) - \widehat{m}_{La}(\infty).$$

The above calculation shows that the second limit in (5.19) is equal to the formally expected value $\widehat{m}_{R\infty}[a,\infty)$ only when $\widehat{m}_{La}(\infty) = 0$, i.e., only when the principal solution $\hat{Y}_a$ is antiprincipal at ∞ according to Remark 3.2(ii).

Equations (5.9) and (5.10) yield the lower bounds

$$m_L(a,\infty] \geq \widehat{m}_{La}(a,\infty], \quad m_R[a,\infty) \geq \widehat{m}_{R\infty}[a,\infty) \tag{5.20}$$

for the numbers of left and right proper focal points of any conjoined basis Y of (H) in the interval $(a,\infty]$ and $[a,\infty)$. Observe that both lower bounds are the same according to (5.17). Observe also, that the second estimate in (5.20) generalizes Proposition 1.2 to possibly uncontrollable system (H). In the next statement we provide the corresponding optimal upper bounds for the numbers $m_L(a,\infty]$ and $m_R[a,\infty)$. These estimates correspond to (2.14) in the case of a compact interval $\mathcal{I} = [a,b]$, see [30, Theorem 5.6].

Theorem 5.7 *(Singular Sturmian separation theorem). Assume that* (1.1) *holds with* $\mathcal{I} = [a,\infty)$ *and system* (H) *is nonoscillatory at* ∞*. Then for any conjoined basis* Y *of* (H) *we have*

$$\widehat{m}_{La}(a,\infty] \leq m_L(a,\infty] \leq \widehat{m}_{L\infty}(a,\infty], \tag{5.21}$$

$$\widehat{m}_{R\infty}[a,\infty) \leq m_R[a,\infty) \leq \widehat{m}_{Ra}[a,\infty). \tag{5.22}$$

Proof. The lower bounds in (5.21) and (5.22) are proven in (5.20). For the upper bound in (5.21) we apply formula (5.1) with $\tilde{Y} := \hat{Y}_\infty$ and $\tilde{D}_\infty := (I,\ 0)^T$ to get

$$m_L(a,\infty] - \widehat{m}_{L\infty}(a,\infty] = \mu\big(\mathcal{J}D_\infty, -E\big) - \mu\big(Y(a), \hat{Y}_\infty(a)\big) = -\mu\big(Y(a), \hat{Y}_\infty(a)\big) \le 0.$$

Similarly, for the upper bound in (5.22) we apply (5.2) with $\tilde{Y} := \hat{Y}_a$ and $\tilde{D}_\infty := \hat{D}^a_\infty$ to get

$$m_R[a,\infty) - \widehat{m}_{Ra}[a,\infty) = \mu^*\big(Y(a), E\big) - \mu^*\big(\mathcal{J}D_\infty, \mathcal{J}\hat{D}^a_\infty\big) = -\mu^*\big(\mathcal{J}D_\infty, \mathcal{J}\hat{D}^a_\infty\big) \le 0.$$

The proof is complete. □

The results in the above theorem yield the following optimal estimates for the left and right proper focal points of any two conjoined bases of (H). They correspond to [30, Corollary 5.8] for the case of a compact interval $\mathcal{I} = [a,b]$.

Corollary 5.8 *(Singular Sturmian separation theorem). Assume that (1.1) holds with $\mathcal{I} = [a,\infty)$ and system (H) is nonoscillatory at ∞. Then for any conjoined bases Y and $\tilde{Y}$ of (H) we have*

$$\begin{aligned} \big| m_L(a,\infty] - \widetilde{m}_L(a,\infty] \big| &\le \operatorname{rank} \hat{X}_\infty(a) \le n, \\ \big| m_R[a,\infty) - \widetilde{m}_R[a,\infty) \big| &\le \operatorname{rank} \hat{X}_\infty(a) \le n, \\ \big| m_L(a,\infty] - \widetilde{m}_R[a,\infty) \big| &\le \operatorname{rank} \hat{X}_\infty(a) \le n. \end{aligned}$$

Proof. The given inequalities follow by the combination of Theorems 5.7 and 5.5. □

Remark 5.9. In [30, Theorem 5.9, Corollary 5.10] we derived some additional Sturmian separation theorems for compact interval $\mathcal{I} = [a,b]$. These results can now be extended to unbounded interval $(a,\infty]$ and $[a,\infty)$ by the methods of this paper. In particular, these results hold also on the corresponding unbounded intervals when the quantities $\operatorname{rank} \hat{X}_a(b)$, $\operatorname{rank} X(b)$, and $\operatorname{rank} \tilde{X}(b)$ are replaced respectively by the quantities $\operatorname{rank} W(\hat{Y}_\infty, \hat{Y}_a)$, $\operatorname{rank} W(\hat{Y}_\infty, Y)$, and $\operatorname{rank} W(\hat{Y}_\infty, \tilde{Y})$.

In the last part of this section we will analyze the numbers $m_L(a,\infty)$ and $m_R(a,\infty)$ of left and right proper focal points of a conjoined basis Y of (H) in the open interval (a,∞). The aim is to find optimal lower and upper bounds for these numbers resembling the results in (5.21) and (5.22) in Theorem 5.7. The motivation for studying the above problem comes from the question whether the estimates on the bounded interval $[a,b]$ in (2.14), or the estimates on the unbounded interval $(a,\infty]$ in (5.21) and on $[a,\infty)$ in (5.22), lead to the conclusions that

$$\widehat{m}_{La}(a,\infty) \le m_L(a,\infty) \le \widehat{m}_{L\infty}(a,\infty), \tag{5.23}$$

$$\widehat{m}_{R\infty}(a,\infty) \le m_R(a,\infty) \le \widehat{m}_{Ra}(a,\infty). \tag{5.24}$$

We will show that the lower bounds in (5.23) and (5.24) are correct and optimal, but the upper bounds in (5.23) and (5.24) are wrong in general, see Remark 5.12 below. In fact, it is surprising that the correct and optimal upper bounds for $m_L(a,\infty)$ and $m_R(a,\infty)$ are the same as in (5.21) and (5.22), i.e., they are equal to $\widehat{m}_{L\infty}(a,\infty]$ and $\widehat{m}_{Ra}[a,\infty)$, respectively.

Another motivation for the study of $m_L(a,\infty)$ comes from the fact that $m_L(a,\infty) = m_L(a,\infty]$ when Y is an antiprincipal solution of (H) at ∞, see Remark 3.2(ii). Examples of such antiprincipal solutions at ∞ are discussed in Proposition 2.11, in Appendix A below, and in [27, Proposition 5.15] and [31, Theorem 7.1].

Theorem 5.10. *Assume that (1.1) holds with $\mathcal{I} = [a, \infty)$ and system (H) is nonoscillatory at ∞. Then for any conjoined basis Y of (H) we have*

$$\widehat{m}_{La}(a,\infty) \le m_L(a,\infty) \le \widehat{m}_{La}(a,\infty) + \operatorname{rank} \mathcal{G}_a^\infty = \widehat{m}_{L\infty}(a,\infty], \tag{5.25}$$

$$\widehat{m}_{R\infty}(a,\infty) \le m_R(a,\infty) \le \widehat{m}_{R\infty}(a,\infty) + \operatorname{rank} \hat{X}_\infty(a^+) = \widehat{m}_{Ra}[a,\infty), \tag{5.26}$$

where $\mathcal{G}_a^\infty$ is the genus of the principal solution $\hat{Y}_a$ near ∞.

Proof. Let Y be a conjoined basis of (H). By (2.12) and (2.14) with $b := t$ we have

$$\widehat{m}_{La}(a,t] \le m_L(a,t] \le \widehat{m}_{La}(a,t] + \operatorname{rank} \hat{X}_a(t) \quad \text{for all } t \in (a,\infty). \tag{5.27}$$

Upon taking the limit as $t \to \infty$ in (5.27) we obtain

$$\widehat{m}_{La}(a,\infty) \le m_L(a,\infty) \le \widehat{m}_{La}(a,\infty) + \operatorname{rank} \mathcal{G}_a^\infty.$$

It remains to prove the last equality in (5.25), for which we utilize the results in Theorems 5.5 and 3.3. In particular, we have

$$\begin{aligned}
\widehat{m}_{L\infty}(a,\infty] &\overset{(5.16)}{=} \widehat{m}_{La}(a,\infty] + \operatorname{rank} \hat{X}_\infty(a) = \widehat{m}_{La}(a,\infty) + \widehat{m}_{La}(\infty) + \operatorname{rank} \hat{X}_\infty(a) \\
&\overset{(3.4)}{=} \widehat{m}_{La}(a,\infty) + \operatorname{rank} \mathcal{G}_a^\infty - \operatorname{rank} W(\hat{Y}_\infty, \hat{Y}_a) + \operatorname{rank} \hat{X}_\infty(a) \\
&= \widehat{m}_{La}(a,\infty) + \operatorname{rank} \mathcal{G}_a^\infty,
\end{aligned}$$

because $W(\hat{Y}_\infty, \hat{Y}_a) = \hat{X}_\infty^T(a)$. Therefore, (5.25) is established. For (5.26) we have by (5.22) with $a := t$ and by the second equation in (5.16) with $a := t$ that

$$\widehat{m}_{R\infty}[t,\infty) \le m_R[t,\infty) \le \widehat{m}_{R\infty}[t,\infty) + \operatorname{rank} \hat{X}_\infty(t) \quad \text{for all } t \in [a,\infty). \tag{5.28}$$

Upon taking the limit as $t \to a^+$ in (5.28) we obtain

$$\widehat{m}_{R\infty}(a,\infty) \le m_R(a,\infty) \le \widehat{m}_{R\infty}(a,\infty) + \operatorname{rank} \hat{X}_\infty(a^+).$$

Finally, the last equality in (5.26) follows from the calculation

$$\begin{aligned}
\widehat{m}_{Ra}[a,\infty) &\overset{(5.16)}{=} \widehat{m}_{R\infty}[a,\infty) + \operatorname{rank} \hat{X}_\infty(a) = \widehat{m}_{R\infty}(a,\infty) + \widehat{m}_{R\infty}(a) + \operatorname{rank} \hat{X}_\infty(a) \\
&\overset{(1.3)}{=} \widehat{m}_{R\infty}(a,\infty) + \operatorname{rank} \hat{X}_\infty(a^+).
\end{aligned}$$

The proof is complete. □

We note that according to Remark 2.6 we can replace in (5.26) the quantity $\operatorname{rank} \hat{X}_\infty(a^+)$ by $\operatorname{rank} \mathcal{G}_\infty(a^+)$, where $\mathcal{G}_\infty(a^+)$ is the genus of $\hat{Y}_\infty$ in the right neighborhood of a. This shows that the estimates in (5.25) and (5.26) are symmetric in some sense.

Remark 5.11. (i) The lower and upper bounds in (5.25) are optimal in the sense that the lower bound is obviously attained for $Y := \hat{Y}_a$, while the upper bound is attained for $Y := \hat{Y}_b$ for b large enough, i.e., $\widehat{m}_{Lb}(a,\infty) = \widehat{m}_{L\infty}(a,\infty]$ for b large enough. We will show that the latter equality holds for all $b > \hat{\alpha}_a$, where

$$\hat{\alpha}_a := \inf\big\{\alpha \in [a,\infty),\ \hat{Y}_a \text{ has constant kernel on the interval } [\alpha,\infty)\big\}.$$

Let $b > \hat{\alpha}_a$. With this choice of b the conjoined basis $\hat{Y}_a$ has constant kernel on the interval $[b,\infty)$, so that $\widehat{m}_{La}(b,\infty) = 0$. Then by the lower bound in (5.25) on the interval (b,∞) and with $Y := \hat{Y}_a$ we obtain that $\widehat{m}_{Lb}(b,\infty) \le \widehat{m}_{La}(b,\infty) = 0$. Hence, $\widehat{m}_{Lb}(b,\infty) = 0$ as well. Then

$$\begin{aligned} \widehat{m}_{Lb}(a,\infty) = \widehat{m}_{Lb}(a,b] + \widehat{m}_{Lb}(b,\infty) &\overset{(2.12)}{=} \widehat{m}_{La}(a,b] + \operatorname{rank} \hat{X}_a(b) \\ &= \widehat{m}_{La}(a,\infty) + \operatorname{rank} \mathcal{G}_a^\infty \overset{(5.25)}{=} \widehat{m}_{L\infty}(a,\infty]. \end{aligned}$$

(ii) In a similar way we show that the lower and upper bounds in (5.26) are also optimal. In particular, the lower bound in (5.26) is attained for $Y := \hat{Y}_\infty$ and the upper bound is attained for $Y := \hat{Y}_b$ for b sufficiently close to a, i.e., $\widehat{m}_{Rb}(a,\infty) = \widehat{m}_{Ra}[a,\infty)$ for b sufficiently close to a. Namely, we will prove that the latter equality holds for all $b \in (a, \hat{\beta}_\infty)$, where

$$\hat{\beta}_\infty := \sup\big\{\beta \in (a,\infty),\ \hat{Y}_\infty \text{ has constant kernel on the interval } (a,\beta]\big\}. \tag{5.29}$$

Let us fix $b \in (a, \hat{\beta}_\infty)$. Then we have by the second formula in (5.17) with $a := b$ that

$$\widehat{m}_{Rb}(a,\infty) = \widehat{m}_{Rb}(a,b) + \widehat{m}_{Rb}[b,\infty) \overset{(5.17)}{=} \widehat{m}_{Rb}(a,b) + \widehat{m}_{L\infty}(b,\infty]. \tag{5.30}$$

Now we show that $\widehat{m}_{Rb}(a,b) = 0$. By (5.29) we know that $\hat{Y}_\infty$ has constant kernel on the interval $[t,b)$ for every $t \in (a,b)$, i.e., $\widehat{m}_{R\infty}[t,b) = 0$ for all $t \in (a,b)$. Then by (2.14) with $a := t$ and $Y := \hat{Y}_\infty$ we conclude that $\widehat{m}_{Rb}[t,b) = 0$ for all $t \in (a,b)$. Upon taking the limit as $t \to a^+$ we obtain that $\widehat{m}_{Rb}(a,b) = \lim_{t\to a^+} \widehat{m}_{Rb}[t,b) = 0$. Next, since $\hat{Y}_\infty$ also satisfies $\widehat{m}_{L\infty}(a,b] = 0$ by (5.29), it follows that $\widehat{m}_{L\infty}(b,\infty] = \widehat{m}_{L\infty}(a,\infty]$, while the latter quantity is equal to $\widehat{m}_{Ra}[a,\infty)$ by (5.17). Therefore, it follows from (5.30) that $\widehat{m}_{Rb}(a,\infty) = \widehat{m}_{Ra}[a,\infty)$.

Remark 5.12. (i) Since by Remark 3.2(ii) the multiplicity of the focal point of $\hat{Y}_\infty$ at ∞ is $\widehat{m}_{L\infty}(\infty) = n - d_\infty$, it follows that $\widehat{m}_{L\infty}(a,\infty] = \widehat{m}_{L\infty}(a,\infty) + n - d_\infty$. This means that the wrong upper bound from (5.23) has to be adjusted (i.e., increased) by the correction term $n - d_\infty$. This shows that this correction term does not depend on the left endpoint a and that the maximal number of left proper focal points of Y in (a,∞) or in $(a,\infty]$ is always greater or equal to $n - d_\infty$. Moreover, the upper bound in (5.23) is indeed optimal only when $d_\infty = n$. In the latter case every conjoined basis of (H) satisfies $m_L(\infty) = 0$. Similarly, the wrong upper bound in (5.24) has to be increased to the number $\widehat{m}_{Ra}[a,\infty)$ by the correction term $\widehat{m}_{Ra}(a) = n - d_a^+$, where d_a^+ is the maximal order of abnormality of (H) in the right neighborhood of a, see [31, Equation (2.17)].

(ii) In the proof of Theorem 5.10 we showed that the upper bound in (5.25) is attained by the principal solution $\hat{Y}_b$ when $b > \hat{\alpha}_a$, i.e., $\widehat{m}_{Lb}(a,\infty) = \widehat{m}_{L\infty}(a,\infty]$. This means that also $\widehat{m}_{Lb}(a,\infty] = \widehat{m}_{L\infty}(a,\infty]$, and consequently $\widehat{m}_{Lb}(\infty) = 0$. In other words, $\hat{Y}_b$ is an antiprincipal solution at ∞ for every $b > \hat{\alpha}_a$. This observation is in agreement with [27, Proposition 5.15] for large b.

(iii) By similar arguments as in part (ii) adjusted to the upper bounds in (5.26) and (5.22) we obtain that for all $b \in (a, \hat{\beta}_\infty)$ the conjoined basis $\hat{Y}_b$ satisfies $\widehat{m}_{Rb}(a) = 0$.

As an analogy of Corollary 5.8 we obtain from Theorem 5.10 the following.

Corollary 5.13. *Assume that* (1.1) *holds with* $\mathcal{I} = [a, \infty)$ *and system* (H) *is nonoscillatory at* ∞. *Then for any two conjoined bases* Y *and* $\tilde{Y}$ *of* (H) *we have*

$$\left| m_L(a,\infty) - \widetilde{m}_L(a,\infty) \right| \le \operatorname{rank} \mathcal{G}_a^\infty \le n, \tag{5.31}$$

$$\left| m_R(a,\infty) - \widetilde{m}_R(a,\infty) \right| \le \operatorname{rank} \hat{X}_\infty(a^+) \le n. \tag{5.32}$$

Proof. The result in (5.31), resp. in (5.32), follows from estimate (5.25), resp. from estimate (5.26), applied to the two conjoined bases Y and $\tilde{Y}$. □

Our final result in this section connects the multiplicities of left and right proper focal points of the principal solutions $\hat{Y}_a$ and $\hat{Y}_\infty$ in the open interval (a, ∞).

Corollary 5.14. *Assume that* (1.1) *holds with* $\mathcal{I} = [a, \infty)$ *and system* (H) *is nonoscillatory at* ∞. *Then we have the equality*

$$\widehat{m}_{La}(a,\infty) + \operatorname{rank} \mathcal{G}_a^\infty = \widehat{m}_{R\infty}(a,\infty) + \operatorname{rank} \hat{X}_\infty(a^+), \tag{5.33}$$

where $\mathcal{G}_a^\infty$ *is the genus of the principal solution* $\hat{Y}_a$ *near* ∞. *In particular, the equality*

$$\widehat{m}_{La}(a,\infty) = \widehat{m}_{R\infty}(a,\infty) \tag{5.34}$$

holds if and only if $\operatorname{rank} \mathcal{G}_a^\infty = \operatorname{rank} \hat{X}_\infty(a^+)$.

Proof. By (3.4) and (1.3) we have

$$\widehat{m}_{La}(\infty) = \operatorname{rank} \mathcal{G}_a^\infty - \operatorname{rank} W(\hat{Y}_\infty, \hat{Y}_a) = \operatorname{rank} \mathcal{G}_a^\infty - \operatorname{rank} \hat{X}_\infty(a),$$

$$\widehat{m}_{R\infty}(a) = \operatorname{rank} \hat{X}_\infty(a^+) - \operatorname{rank} \hat{X}_\infty(a).$$

Therefore, equation (5.33) is equivalent with the first equality in (5.17). The second statement in the corollary then follows directly from (5.33). □

In the completely controllable case every conjoined basis Y of (H) has $X(t)$ invertible near a and near ∞. Therefore, the condition $\operatorname{rank} \mathcal{G}_a^\infty = \operatorname{rank} \hat{X}_\infty(a^+) = n$ is automatically satisfied and we get from Corollary 5.14 the following. This result is also new even in this special setting.

Corollary 5.15. *Assume that* (1.1) *holds with* $\mathcal{I} = [a, \infty)$ *and system* (H) *is completely controllable on* $[a, \infty)$ *and nonoscillatory at* ∞. *Then the principal solutions* $\hat{Y}_a$ *and* $\hat{Y}_\infty$ *satisfy the equality in* (5.34), *i.e.,* $\hat{Y}_a$ *and* $\hat{Y}_\infty$ *have the same number of focal points in* (a, ∞).

Remark 5.16. It is easy to see that under assumption (1.1) with $\mathcal{I} = (-\infty, b]$ and for a nonoscillatory system (H) at $-\infty$ the results in this section hold also for the numbers of left and right proper focal points of the conjoined bases Y, $\hat{Y}_{-\infty}$, and $\hat{Y}_b$ in the intervals $(-\infty, b]$ and $[-\infty, b)$, respectively in the open interval $(-\infty, b)$.

6. Asymptotic properties of comparative index

In this section we apply the results in Section 5 to obtain asymptotic formulas for the comparative indices $\mu\big(Y(t), \tilde{Y}(t)\big)$ and $\mu^*\big(Y(t), \tilde{Y}(t)\big)$ when $t \to \pm\infty$. Moreover, in addition to (3.4) and (3.14) we derive another representation formulas for the multiplicities $m_L(\infty)$ and $m_R(-\infty)$ for a conjoined basis Y in terms of limits at $\pm\infty$ of comparative indices involving Y and the principal solution $\hat{Y}_t$. These results essentially extend the limit properties of the comparative index in (2.19) or in [30, Section 6] to the case of $t_0 = \pm\infty$.

Theorem 6.1. *Assume that (1.1) holds with $\mathcal{I} = [a, \infty)$ and system (H) is nonoscillatory at ∞. Then for any two conjoined bases Y and $\tilde{Y}$ of (H) the limits of the comparative indices $\mu\big(Y(t), \tilde{Y}(t)\big)$ and $\mu^*\big(Y(t), \tilde{Y}(t)\big)$ for $t \to \infty$ exist and*

$$\mu_\infty(Y, \tilde{Y}) := \lim_{t\to\infty} \mu\big(Y(t), \tilde{Y}(t)\big) = \mu(\mathcal{J}D_\infty, \mathcal{J}\tilde{D}_\infty) - m_L(\infty) + \widetilde{m}_L(\infty), \tag{6.1}$$

$$\mu^*_\infty(Y, \tilde{Y}) := \lim_{t\to\infty} \mu^*\big(Y(t), \tilde{Y}(t)\big) = \mu^*(\mathcal{J}D_\infty, \mathcal{J}\tilde{D}_\infty), \tag{6.2}$$

where D_∞ and $\tilde{D}_\infty$ are the constant matrices in (4.4) corresponding to Y and $\tilde{Y}$.

Proof. By Theorem 5.1 we know that for every $t \in [a, \infty)$

$$\mu\big(Y(t), \tilde{Y}(t)\big) = \mu(\mathcal{J}D_\infty, \mathcal{J}\tilde{D}_\infty) - m_L(t, \infty] + \widetilde{m}_L(t, \infty], \tag{6.3}$$

$$\mu^*\big(Y(t), \tilde{Y}(t)\big) = \mu^*(\mathcal{J}D_\infty, \mathcal{J}\tilde{D}_\infty) + m_R[t, \infty) - \widetilde{m}_R[t, \infty). \tag{6.4}$$

Since system (H) is nonoscillatory at ∞, the conjoined bases Y and $\tilde{Y}$ have no left and right proper focal points in the intervals (t, ∞) and $[t, \infty)$ for sufficiently large t, i.e., $m_L(t, \infty) = 0 = \widetilde{m}_L(t, \infty)$ and $m_R[t, \infty) = 0 = \widetilde{m}_R[t, \infty)$ for large t. Therefore, upon taking the limit as $t \to \infty$ in (6.3) and (6.4) we obtain (6.1) and (6.2), respectively. □

Remark 6.2. By taking $\tilde{Y} := \hat{Y}_\infty$ we have $\mathcal{J}\tilde{D}_\infty = -E$ and hence, $\mu(\mathcal{J}D_\infty, \mathcal{J}\tilde{D}_\infty) = 0$ by (2.28) and (2.25). Therefore, in this case equation (6.1) with $\widetilde{m}_L(\infty) = \widehat{m}_{L\infty}(\infty) = n - d_\infty$ and equation (6.2) yield

$$\mu_\infty(Y, \hat{Y}_\infty) = n - d_\infty - m_L(\infty) \overset{(3.1)}{=} \operatorname{rank} T_{\alpha,\infty}, \quad \mu^*_\infty(Y, \hat{Y}_\infty) = 0,$$

where $\alpha \in [a, \infty)$ is such that $\hat{Y}_\infty$ has constant kernel on $[\alpha, \infty)$ and $d[\alpha, \infty) = d_\infty$. This reveals another interesting property of the matrix $T_{\alpha,\infty}$ associated with Y, namely that its rank is equal to the limit of the comparative index $\mu(Y(t), \hat{Y}_\infty(t))$ as $t \to \infty$.

Remark 6.3. By [30, Theorem 6.1] or [12, Theorem 2.3] we know that the comparative index $\mu(Y(t), \tilde{Y}(t))$ is right continuous, the dual comparative index $\mu^*(Y(t), \tilde{Y}(t))$ is left continuous, and the formula

$$\lim_{t\to t_0^-} \mu(Y(t), \tilde{Y}(t)) = \mu(Y(t_0), \tilde{Y}(t_0)) - m_L(t_0) + \widetilde{m}_L(t_0)$$

holds. Therefore, the left discontinuity of $\mu(Y(t), \tilde{Y}(t))$ at t_0 measures the difference between the multiplicities $m_L(t_0)$ and $\widetilde{m}_L(t_0)$. From this point of view the results in (6.1) and (6.2) of Theorem 6.1 can be interpreted as a compactification of these properties on the extended interval $[a,\infty] = [a,\infty) \cup \{\infty\}$, where we would define $\mu(Y(\infty), \tilde{Y}(\infty)) := \mu(\mathcal{J}D_\infty, \mathcal{J}\tilde{D}_\infty)$ and $\mu^*(Y(\infty), \tilde{Y}(\infty)) := \mu^*(\mathcal{J}D_\infty, \mathcal{J}\tilde{D}_\infty)$. In this case the left discontinuity of $\mu(Y(t), \tilde{Y}(t))$ at ∞, i.e., formula (6.1), measures the difference between the multiplicities $m_L(\infty)$ and $\widetilde{m}_L(\infty)$.

In the next theorem we present a formula for calculating the multiplicity of the focal point at ∞ of a conjoined basis Y in terms of the comparative index of Y with the principal solution $\hat{Y}_t$, respectively in terms of their representing matrices D_∞ and D_∞^t in (4.4) and (4.6). This result corresponds to [30, Theorems 6.3 and 6.5] in the case of a compact interval $\mathcal{I}$.

Theorem 6.4. *Assume that (1.1) holds with $\mathcal{I} = [a,\infty)$ and system (H) is nonoscillatory at ∞. Then for any conjoined basis Y of (H) the limits of the comparative indices $\mu\big(\mathcal{J}D_\infty, \mathcal{J}D_\infty^t\big)$ and $\mu^*\big(\mathcal{J}D_\infty, \mathcal{J}D_\infty^t\big)$ for $t\to\infty$ exist and*

$$\lim_{t\to\infty} \mu\big(\mathcal{J}D_\infty, \mathcal{J}D_\infty^t\big) = m_L(\infty), \tag{6.5}$$

$$\lim_{t\to\infty} \mu^*\big(\mathcal{J}D_\infty, \mathcal{J}D_\infty^t\big) = n - d_\infty, \tag{6.6}$$

where D_∞ and D_∞^t are the constant matrices in (4.4) and (4.6) corresponding to Y and $\hat{Y}_t$.

Proof. By Corollary 5.3 and Theorem 5.5 we have for every $t \in [a,\infty)$ that

$$\mu\big(\mathcal{J}D_\infty, \mathcal{J}D_\infty^t\big) \overset{(5.9)}{=} m_L(t,\infty] - \widehat{m}_{Lt}(t,\infty] \overset{(5.17)}{=} m_L(t,\infty] - \widehat{m}_{R\infty}[t,\infty). \tag{6.7}$$

Since system (H) is nonoscillatory at ∞, the conjoined bases Y and $\hat{Y}_\infty$ have no left and right proper focal points in the intervals (t,∞) and $[t,\infty)$ for sufficiently large t, i.e., $m_L(t,\infty) = 0 = \widehat{m}_{R\infty}[t,\infty)$ for large t. Therefore, upon taking the limit as $t\to\infty$ in (6.7) we obtain (6.5). Alternatively we can use the first formula in (4.19) (with $\alpha := t$) to obtain (6.5) directly. Next we show that formula (6.6) follows from (6.5) by using the relationship between the dual comparative index and the comparative index in (2.23) and (2.24). We fix $t \in [a,\infty)$. By the form of the matrices $\mathcal{J}D_\infty^t$ in (5.18) with $a := t$ and $\mathcal{J}D_\infty$ in (4.4) we obtain from (2.23) that

$$\mu^*\big(\mathcal{J}D_\infty^t, \mathcal{J}D_\infty\big) + \operatorname{rank} \hat{X}_\infty(t) = \mu\big(\mathcal{J}D_\infty, \mathcal{J}D_\infty^t\big) + \operatorname{rank} W(\hat{Y}_\infty, Y), \tag{6.8}$$

where we also used that $W(\hat{Y}_\infty, \hat{Y}_t) = \hat{X}_\infty^T(t)$. Moreover, from (2.24) we get

$$\mu^*\big(\mathcal{J}D_\infty, \mathcal{J}D_\infty^t\big) + \mu^*\big(\mathcal{J}D_\infty^t, \mathcal{J}D_\infty\big) = \operatorname{rank} W(\mathcal{J}D_\infty, \mathcal{J}D_\infty^t) = \operatorname{rank} X(t), \tag{6.9}$$

where we used that $W(\mathcal{J}D_\infty, \mathcal{J}D^t_\infty) = W(Y, \hat{Y}_t) = X^T(t)$. Upon subtracting (6.8) from equation (6.9) we obtain

$$\mu^*\big(\mathcal{J}D_\infty, \mathcal{J}D^t_\infty\big) = \operatorname{rank} \hat{X}_\infty(t) + \operatorname{rank} X(t) - \operatorname{rank} W(\hat{Y}_\infty, Y) - \mu\big(\mathcal{J}D_\infty, \mathcal{J}D^t_\infty\big). \tag{6.10}$$

If $\mathcal{G}^\infty_{\min}$ and $\mathcal{G}^\infty$ are the genera of conjoined bases corresponding to $\hat{Y}_\infty$ and Y, respectively, then $\operatorname{rank} \hat{X}_\infty(t) \equiv \operatorname{rank} \mathcal{G}^\infty_{\min} = n - d_\infty$ and $\operatorname{rank} X(t) \equiv \operatorname{rank} \mathcal{G}^\infty$ for large t. This fact together with the validity of (6.5) imply that the limit of $\mu^*\big(\mathcal{J}D_\infty, \mathcal{J}D^t_\infty\big)$ as $t \to \infty$ exists, and by (6.10) and (6.5) it is equal to

$$\lim_{t\to\infty} \mu^*\big(\mathcal{J}D_\infty, \mathcal{J}D^t_\infty\big) = n - d_\infty + \operatorname{rank} \mathcal{G}^\infty - \operatorname{rank} W(\hat{Y}_\infty, Y) - m_L(\infty) \overset{(3.4)}{=} n - d_\infty.$$

The proof of formula (6.6) is complete. □

Remark 6.5. If assumption (1.1) holds with $\mathcal{I} = (-\infty, b]$ and system (H) is nonoscillatory at $-\infty$, then analogous results as in Theorems 6.1 and 6.4 and Remark 6.2 hold for the limits of the corresponding comparative indices as $t \to -\infty$. More precisely, we have the formulas

$$\begin{aligned}
\mu_{-\infty}(Y, \tilde{Y}) &:= \lim_{t\to-\infty} \mu\big(Y(t), \tilde{Y}(t)\big) = \mu(\mathcal{J}D_{-\infty}, \mathcal{J}\tilde{D}_{-\infty}),\\
\mu^*_{-\infty}(Y, \tilde{Y}) &:= \lim_{t\to-\infty} \mu^*\big(Y(t), \tilde{Y}(t)\big) = \mu^*(\mathcal{J}D_{-\infty}, \mathcal{J}\tilde{D}_{-\infty}) - m_R(-\infty) + \widetilde{m}_R(-\infty),\\
\mu_{-\infty}(Y, \hat{Y}_{-\infty}) &= 0, \quad \mu^*_{-\infty}(Y, \hat{Y}_{-\infty}) = \operatorname{rank} T_{\beta,\,-\infty},
\end{aligned}$$

where $\beta \in (-\infty, b]$ is such that $\hat{Y}_{-\infty}$ has constant kernel on $(-\infty, \beta]$ and $d(-\infty, \beta] = d_{-\infty}$. Moreover, we have the limits

$$\begin{aligned}
\lim_{t\to-\infty} \mu^*\big(\mathcal{J}D_{-\infty}, \mathcal{J}D^t_{-\infty}\big) &= m_R(-\infty),\\
\lim_{t\to-\infty} \mu\big(\mathcal{J}D_{-\infty}, \mathcal{J}D^t_{-\infty}\big) &= n - d_{-\infty},
\end{aligned}$$

where $D_{-\infty}$, $\tilde{D}_{-\infty}$, and $D^t_{-\infty}$ are the constant matrices corresponding to Y, $\tilde{Y}$, and $\hat{Y}_t$ with respect to the symplectic fundamental matrix $\hat{\Phi}_{-\infty}(t) = \big(\hat{Y}_{-\infty}(t) \quad \bar{Y}_{-\infty}(t)\big)$ involving the minimal principal solution $\hat{Y}_{-\infty}$ of (H) at $-\infty$, i.e., as in (4.4) and (5.8) we have

$$\mathcal{J}D_{-\infty} = \begin{pmatrix} W(\hat{Y}_{-\infty}, Y) \\ W(\bar{Y}_{-\infty}, Y) \end{pmatrix}, \quad \mathcal{J}\tilde{D}_{-\infty} = \begin{pmatrix} W(\hat{Y}_{-\infty}, \tilde{Y}) \\ W(\bar{Y}_{-\infty}, \tilde{Y}) \end{pmatrix}, \quad \mathcal{J}D^t_{-\infty} = \begin{pmatrix} W(\hat{Y}_{-\infty}, \hat{Y}_t) \\ W(\bar{Y}_{-\infty}, \hat{Y}_t) \end{pmatrix}.$$

Remark 6.6. The results in Theorems 5.1 and 6.1 and in Remark 6.5 allow to interpret the limit case of the formulas (2.15) and (2.16) for $b \to \infty$. Indeed, by (5.1) and (6.1), respectively by (5.2) and (6.2), for any conjoined bases Y and $\tilde{Y}$ we have

$$m_L(a,\infty) - \widetilde{m}_L(a,\infty) = \mu_\infty(Y,\tilde{Y}) - \mu(Y(a),\tilde{Y}(a)),$$
$$m_R[a,\infty) - \widetilde{m}_R[a,\infty) = \mu^*(Y(a),\tilde{Y}(a)) - \mu^*_\infty(Y,\tilde{Y}),$$
$$m_L(-\infty,b] - \widetilde{m}_L(-\infty,b] = \mu(Y(b),\tilde{Y}(b)) - \mu_{-\infty}(Y,\tilde{Y}),$$
$$m_R(-\infty,b) - \widetilde{m}_R(-\infty,b) = \mu^*_{-\infty}(Y,\tilde{Y}) - \mu^*(Y(b),\tilde{Y}(b)).$$

These formulas represent the continuous time versions of the limit formulas for conjoined bases of one discrete symplectic system in [7, Theorem 4.1].

7. Sturm–Liouville differential equation

In this section we will discuss some of the results from Section 5 for the second order Sturm–Liouville differential equation

$$\big(r(t)\,y'\big)' + p(t)\,y(t) = 0, \quad t \in [a,\infty), \tag{SL}$$

where $r, p : [a,\infty) \to \mathbb{R}$ are given piecewise continuous functions such that $r(t) > 0$ on $[a,\infty)$. As it is common for piecewise continuous functions, the assumption of the positivity of r means that the one-sided limits of $r(t)$ at its points of discontinuity are also positive, so that $1/r$ is also piecewise continuous on $[a,\infty)$. Under these assumptions equation (SL) is a special case of system (H) with $n = 1$, $A(t) \equiv 0$, $B(t) = 1/r(t)$, and $C(t) = -p(t)$, that is, $x := y$ and $u := r(t)\,y'$. For classical results (oscillation, nonoscillation, disconjugacy) about equation (SL) with continuous coefficients on $[a,\infty)$ we refer to [5,16,21,22]. We note that equation (SL) or the corresponding system (H) with the above coefficients is completely controllable on $[a,\infty)$.

In order to apply the results from Section 5, we denote by $\hat{y}_a$ the principal solution of (SL) at a, i.e., it is determined by the initial conditions $\hat{y}_a(a) = 0$ and $\hat{y}_a'(a) = 1/r(a)$. We recall that the principal solution $\hat{y}_\infty$ of a nonoscillatory equation (SL) at ∞ is defined by the condition $\int_\alpha^\infty 1/[r(t)\,\hat{y}_\infty^2(t)]\,\mathrm{d}t = \infty$, where $\alpha \in [a,\infty)$ is such that $\hat{y}_\infty(t) \neq 0$ on $[\alpha,\infty)$. The principal solutions $\hat{y}_a$ and $\hat{y}_\infty$ are uniquely determined (up to a nonzero constant multiple).

The zeros (i.e., focal points in the terminology of the previous sections) of a nontrivial solution y of (SL) are defined by the condition $y(t_0) = 0$, when t_0 is a finite point. In addition, according to (3.15) in Remark 3.7 we say that *y has a zero at ∞* if $y = \hat{y}_\infty$, i.e., if y is principal at ∞. All the zeros of y are simple, i.e., their multiplicities are 1. In accordance with (1.5) we will use the unified notation $m(t_0)$, $\widetilde{m}(t_0)$, $\widehat{m}_a(t_0)$, $\widehat{m}_\infty(t_0)$ for the multiplicities of t_0 of the solutions y, $\tilde{y}$, $\hat{y}_a$, $\hat{y}_\infty$. That is, for a solution y of (SL) we denote $m(t_0) := m_L(t_0)$ for $t_0 \in (a,\infty]$ or $m(t_0) := m_R(t_0)$ for $t_0 \in [a,\infty)$, where the equality $m_L(t_0) = m_R(t_0)$ holds for every $t_0 \in (a,\infty)$. Similar notation will be used for the numbers of zeros of y, $\tilde{y}$, $\hat{y}_a$, $\hat{y}_\infty$ in some given interval. In this context $m(\infty) = 0$ if and only if y is an antiprincipal solution at ∞, or in other words $\int_\alpha^\infty 1/[r(t)\,y^2(t)]\,\mathrm{d}t < \infty$ for some $\alpha \in [a,\infty)$ such that $y(t) \neq 0$ on $[\alpha,\infty)$.

Remark 7.1. Recently we have observed the paper [1], which studies the validity of the singular Sturmian comparison theorem for two equations of the form (SL) under a standard (strict) majorant condition. However, the approach in this reference does not allow to apply the results to one equation (SL), hence to deduce the corresponding singular Sturmian separation theorem. As a consequence of Theorems 5.5 and 5.7 we obtain the corresponding statement as follows. Assume that equation (SL) is nonoscillatory at ∞ and let y_1 and y_2 be two linearly independent solutions of (SL). If y_1 has two consecutive zeros at $t_1, t_2 \in [a, \infty]$, $t_1 < t_2$, then y_2 has exactly one zero in the open interval (t_1, t_2).

Another motivation for the study of equation (SL) we found in the paper [6], which uses the principal solution $\hat{y}_\infty$ to derive some existence theorems for second order nonlinear differential equations. In particular, in [6, Remark 1, Lemma 5] the authors study the question when

$$\hat{y}_\infty(t) \neq 0 \ \text{ on the whole interval } [a, \infty) \tag{7.1}$$

and relate this condition to the disconjugacy of (SL) on the interval $[a, \infty)$. Following the classical terminology, see [16, Section XI.6] or [5], we define equation (SL) to be *disconjugate* on an interval $\mathcal{I}_0 \subseteq [a, \infty]$ if any nontrivial solution of (SL) has at most one zero in $\mathcal{I}_0$. Note that the right endpoint ∞ is now included in the above definition.

It is well known that the disconjugacy of (SL) on $[a, \infty)$ is a condition, which is necessary but not sufficient for the validity of (7.1). This fact is illustrated by [6, Example 1], where (SL) is disconjugate on $[a, \infty)$ but the principal solution $\hat{y}_\infty$ satisfies $\hat{y}_\infty(t) > 0$ on (a, ∞) and $\hat{y}_\infty(a) = 0$, i.e., (7.1) does not hold. This clearly means that $\hat{y}_\infty = \hat{y}_a$ (up to a nonzero constant multiple) and this solution has two zeros in the interval $[a, \infty]$. Hence, equation (SL) from [6, Example 1] is not disconjugate on $[a, \infty]$ in the present setting.

The following characterization of (7.1) is an immediate consequence of the above definition and the results in Theorems 5.5 and 5.7. We can see that the formulation is in the same spirit as in the regular case, see [5, Theorems 1.1 and 1.2].

Theorem 7.2. *The following statements are equivalent.*

(i) *Condition* (7.1) *holds.*
(ii) *The solution $\hat{y}_a$ is positive on (a, ∞) and antiprincipal at ∞.*
(iii) *Equation* (SL) *is disconjugate on the interval $[a, \infty]$.*
(iv) *There exists a solution of* (SL) *with no zeros in the interval $[a, \infty]$.*

Proof. Assume that (i) holds, i.e., $\widehat{m}_\infty[a, \infty) = 0$. Then by (5.17) we get $\widehat{m}_a(a, \infty] = 0$, so that $\hat{y}_a(t) \neq 0$ (positive) on (a, ∞), as well as $\widehat{m}_a(\infty) = 0$. The latter condition means that $\hat{y}_a$ is not principal at ∞, i.e., (ii) holds. Next we assume that (ii) is satisfied, that is, $\widehat{m}_a(a) = 1$ and $\widehat{m}_a(a, \infty] = 0$. Then $\widehat{m}_a(\infty) = 0$ and $\widehat{m}_a[a, \infty) = 1$. By (5.15) we get $\operatorname{rank} \hat{y}_\infty(a) = 1$, so that by (5.16) we have $\widehat{m}_\infty(a, \infty] = 1$. This implies through (5.21) that $m(a, \infty] \leq 1$ for every nontrivial solution y of (SL). If equation (SL) has a solution $\tilde{y}$ with two zeros $t_1 < t_2$ in $[a, \infty]$, then $\tilde{y}$ is not a constant multiple of $\hat{y}_a$, and hence $a < t_1$. Thus, $\widetilde{m}(a, \infty] = 2$, which is a contradiction. Therefore, every nontrivial solution of (SL) has at most one zero in $[a, \infty]$, i.e., (iii) holds. Next we assume that (iii) is satisfied. Then the solution $\hat{y}_a$ satisfies $\widehat{m}_a(a) = 1$ and $\widehat{m}_a(a, \infty] = 0$. Hence, $\hat{y}_a$ is not principal at ∞. Since the disconjugacy of (SL) on $[a, \infty]$ implies the nonoscillation of (SL) at ∞, the principal solution $\hat{y}_\infty$ exists and satisfies $\widehat{m}_\infty[a, \infty) = 0$, compare with (5.17). Then without loss of generality $\hat{y}_a(t) > 0$ on (a, ∞) and $\hat{y}_\infty(t) > 0$ on $[a, \infty)$. Also, since $\widehat{m}_\infty(\infty) = 1$, the solutions $\hat{y}_a$ and $\hat{y}_\infty$ are linearly independent. Then the solution $\tilde{y} := \hat{y}_a + \hat{y}_\infty$ satisfies $\tilde{y}(t) > 0$ on $[a, \infty)$ and $\tilde{y}$ is not principal at ∞, i.e., $\widetilde{m}(\infty) = 0$. Hence, $\tilde{y}$ has no zeros in $[a, \infty]$. Finally, if (iv) is satisfied and $\tilde{y}$ is the solution with no zeros in $[a, \infty]$, then by (5.22) with $y := \tilde{y}$ we have $\widehat{m}_\infty[a, \infty) \leq \widetilde{m}[a, \infty) = 0$. This means that $\hat{y}_\infty$ has no zeros in $[a, \infty)$, i.e., (7.1) holds. The proof is complete. $\square$

The results in Remark 7.1 and Theorem 7.2 clearly hold also on the unbounded intervals $[-\infty, b]$ or $[-\infty, \infty]$, see also Remark 8.1, which we illustrate by the following example.

Example 7.3. Equation $y'' = 0$ has the principal solution $\hat{y}_{\infty} = \hat{y}_{-\infty} \equiv 1$ at $\pm\infty$, which is positive on $(-\infty, \infty)$ but has two zeros in the interval $[-\infty, \infty]$ (its zeros are at $\pm\infty$). Therefore, this equation is disconjugate on $(-\infty, \infty)$, $(-\infty, \infty]$, or $[-\infty, \infty)$, but it is not disconjugate on the interval $[-\infty, \infty]$. Moreover, according to Remark 7.1 every solution y_2, which is linearly independent with $y_1 \equiv 1$, has exactly one zero in the open interval $(-\infty, \infty)$. This obviously holds, since $y_2(t) = kt + q$ for $k, q \in \mathbb{R}$ with $k \neq 0$.

8. Conclusions and remarks

In this paper we have presented Sturmian separation theorems for possibly uncontrollable linear Hamiltonian systems (H) on unbounded intervals $[a, \infty)$ or $(-\infty, b]$. We showed that the classical Sturmian separation theorems on a compact interval $[a, b]$ obtained recently in [12,30] can be extended to unbounded intervals by using suitable properties of the comparative index. The key feature is to define properly the multiplicities of proper focal points at ∞ and $-\infty$ (Definition 3.1, Theorem 3.3, and Remark 3.6) and to count the multiplicities of proper focal points including those at ∞ (for the left proper focal points) and at $-\infty$ (for the right proper focal points). The main results contain exact formulas for the difference of the numbers of proper focal points of two conjoined bases Y and $\tilde{Y}$ (Theorem 5.1), exact formulas for the numbers of proper focal points of one conjoined basis Y (Corollary 5.3), the relationship between the numbers of left and right proper focal points for a given conjoined basis Y (Theorem 5.4), comparison of the numbers of proper focal points of the (minimal) principal solutions $\hat{Y}_a$ and $\hat{Y}_{\infty}$ (Theorem 5.5), optimal lower and upper bounds for numbers of proper focal points of any conjoined basis Y (Theorems 5.7 and 5.10), optimal estimates for the difference of proper focal points of any two conjoined bases Y and $\tilde{Y}$ (Corollaries 5.8 and 5.13), and asymptotic formulas for the comparative index and the dual comparative index involving two conjoined bases Y and $\tilde{Y}$ (Theorems 6.1 and 6.4). Also, in the appendix below we derived new characterizations of antiprincipal solutions of (H) at ∞ in terms of a Wronskian (Theorem A.1) and in terms of a limit (Theorem A.4), which are used in the proof of Theorem 3.3 to calculate the multiplicities of focal points at ∞.

We emphasize that all the results in this paper are new even for a completely controllable linear Hamiltonian system (H).

We are convinced that the contributions in this paper will motivate further development of the oscillation theory for linear Hamiltonian systems and Sturm–Liouville differential equations. For example, unified Sturmian separation theorems on regular intervals (i.e., compact intervals) and on singular type intervals can be obtained as an immediate consequence of the results in [30] and those in this paper. We will provide detailed statements of this unified theory in a separate note. Furthermore, the new notion of a multiplicity of a focal point at $\pm\infty$ (Definition 3.1 and Remark 3.6) will lead to a new interpretation of other classical topics in the theory of differential equations, such as the disconjugacy of system (H) on unbounded intervals $[a, \infty)$, $[a, \infty]$, or $[-\infty, \infty]$. This topic will also be addressed in our subsequent work.

Remark 8.1. In this paper we considered linear Hamiltonian system (H) on the unbounded intervals of the form $[a, \infty)$ or $(-\infty, b]$, i.e., for intervals with one singular endpoint. The formulations of the main results discussed above show that the presented theory remains valid also for the interval $\mathcal{I}$ with two singular endpoints, i.e., for $a = -\infty$ and $\mathcal{I} = \mathbb{R} = (-\infty, \infty)$. This can be seen by splitting the interval $\mathcal{I}$ into two disjoint intervals $(-\infty, b] \cup (a, \infty)$ with $b = a$ and by using that for any conjoined basis Y we have

$$m_L(-\infty, \infty] = m_L(-\infty, b] + m_L(a, \infty], \quad m_R[-\infty, \infty) = m_R[-\infty, b) + m_R[a, \infty).$$

In this case the quantities $Y(a)$, $\tilde{Y}(a)$, $\hat{Y}_\infty(a)$ and $\operatorname{rank} X(a)$, $\operatorname{rank} \hat{X}_\infty(a)$, $\operatorname{rank} \mathcal{G}_a^\infty$ should be replaced by the quantities $\mathcal{J}D_{-\infty}$, $\mathcal{J}\tilde{D}_{-\infty}$, $\mathcal{J}D_{-\infty}^{\infty}$ and $\operatorname{rank} W(\hat{Y}_{-\infty}, Y)$, $\operatorname{rank} W(\hat{Y}_{-\infty}, \hat{Y}_\infty)$, $\operatorname{rank} \mathcal{G}_{-\infty}^{\infty}$, respectively, compare with Remarks 5.2 and 5.16. More precisely, for a system (H) which satisfies the Legendre condition (1.1) with $\mathcal{I} = \mathbb{R}$ and which is nonoscillatory both at ∞ and at $-\infty$, we have from Theorem 5.1 for any conjoined bases Y and $\tilde{Y}$ of (H) the equalities

$$m_L(-\infty, \infty] - \widetilde{m}_L(-\infty, \infty] = \mu\big(\mathcal{J}D_\infty, \mathcal{J}\tilde{D}_\infty\big) - \mu\big(\mathcal{J}D_{-\infty}, \mathcal{J}\tilde{D}_{-\infty}\big), \tag{8.1}$$

$$m_R[-\infty, \infty) - \widetilde{m}_R[-\infty, \infty) = \mu^*\big(\mathcal{J}D_{-\infty}, \mathcal{J}\tilde{D}_{-\infty}\big) - \mu^*\big(\mathcal{J}D_\infty, \mathcal{J}\tilde{D}_\infty\big), \tag{8.2}$$

from Corollary 5.3 for any conjoined basis Y of (H) the equalities

$$m_L(-\infty, \infty] = \widehat{m}_{L-\infty}(-\infty, \infty] + \mu\big(\mathcal{J}D_\infty, \mathcal{J}D_\infty^{-\infty}\big), \tag{8.3}$$

$$m_R[-\infty, \infty) = \widehat{m}_{R\infty}[-\infty, \infty) + \mu^*\big(\mathcal{J}D_{-\infty}, \mathcal{J}D_{-\infty}^{\infty}\big), \tag{8.4}$$

from Theorem 5.4 for any conjoined basis Y of (H) the equality

$$m_L(-\infty, \infty] + \operatorname{rank} W(\hat{Y}_\infty, Y) = m_R[-\infty, \infty) + \operatorname{rank} W(\hat{Y}_{-\infty}, Y), \tag{8.5}$$

from Theorem 5.5 the equalities

$$\left.\begin{aligned} \widehat{m}_{L\infty}(-\infty, \infty] &= \widehat{m}_{L-\infty}(-\infty, \infty] + \operatorname{rank} W(\hat{Y}_\infty, \hat{Y}_{-\infty}), \\ \widehat{m}_{R-\infty}[-\infty, \infty) &= \widehat{m}_{R\infty}[-\infty, \infty) + \operatorname{rank} W(\hat{Y}_\infty, \hat{Y}_{-\infty}), \\ \widehat{m}_{L-\infty}(-\infty, \infty] &= \widehat{m}_{R\infty}[-\infty, \infty), \\ \widehat{m}_{R-\infty}[-\infty, \infty) &= \widehat{m}_{L\infty}(-\infty, \infty], \end{aligned}\right\} \tag{8.6}$$

from Theorems 5.7 and 5.10 for any conjoined basis Y of (H) the optimal estimates

$$\left.\begin{aligned} \widehat{m}_{L-\infty}(-\infty, \infty] \le m_L(-\infty, \infty] &\le \widehat{m}_{L\infty}(-\infty, \infty], \\ \widehat{m}_{R\infty}[-\infty, \infty) \le m_R[-\infty, \infty) &\le \widehat{m}_{R-\infty}[-\infty, \infty), \\ \widehat{m}_{L-\infty}(-\infty, \infty) \le m_L(-\infty, \infty) &\le \widehat{m}_{L-\infty}(-\infty, \infty) + \operatorname{rank} \mathcal{G}_{-\infty}^{\infty} = \widehat{m}_{L\infty}(-\infty, \infty], \\ \widehat{m}_{R\infty}(-\infty, \infty) \le m_R(-\infty, \infty) &\le \widehat{m}_{R\infty}(-\infty, \infty) + \operatorname{rank} \mathcal{G}_{\infty}^{-\infty} = \widehat{m}_{R-\infty}[-\infty, \infty), \end{aligned}\right\} \tag{8.7}$$

and from Corollaries 5.8 and 5.13 for any conjoined bases Y and $\tilde{Y}$ of (H) the optimal estimates

$$\left.\begin{aligned} \big|m_L(-\infty, \infty] - \widetilde{m}_L(-\infty, \infty]\big| &\le \operatorname{rank} W(\hat{Y}_\infty, \hat{Y}_{-\infty}) \le n, \\ \big|m_R[-\infty, \infty) - \widetilde{m}_R[-\infty, \infty)\big| &\le \operatorname{rank} W(\hat{Y}_\infty, \hat{Y}_{-\infty}) \le n, \\ \big|m_L(-\infty, \infty] - \widetilde{m}_R[-\infty, \infty)\big| &\le \operatorname{rank} W(\hat{Y}_\infty, \hat{Y}_{-\infty}) \le n, \\ \big|m_L(-\infty, \infty) - \widetilde{m}_L(-\infty, \infty)\big| &\le \operatorname{rank} \mathcal{G}_{-\infty}^{\infty} \le n, \\ \big|m_R(-\infty, \infty) - \widetilde{m}_R(-\infty, \infty)\big| &\le \operatorname{rank} \mathcal{G}_{\infty}^{-\infty} \le n, \end{aligned}\right\} \tag{8.8}$$

where $\mathcal{G}^{\infty}_{-\infty}$ is the genus of $\hat{Y}_{-\infty}$ near ∞ and $\mathcal{G}^{-\infty}_{\infty}$ is the genus of $\hat{Y}_{\infty}$ near $-\infty$. We note that the upper bounds in (8.8) satisfy

$$\operatorname{rank} W(\hat{Y}_{\infty}, \hat{Y}_{-\infty}) \le \min\left\{\operatorname{rank} \mathcal{G}^{\infty}_{-\infty},\ \operatorname{rank} \mathcal{G}^{-\infty}_{\infty}\right\},$$

which follows from (3.4) with $Y := \hat{Y}_{-\infty}$, from (3.14) with $Y := \hat{Y}_{\infty}$, and from the equality $W(\hat{Y}_{-\infty}, \hat{Y}_{\infty}) = -[W(\hat{Y}_{\infty}, \hat{Y}_{-\infty})]^T$. A simple illustration of these estimates is presented in Example 7.3. Moreover, from Corollary 5.14 we have

$$\widehat{m}_{L-\infty}(-\infty,\infty) + \operatorname{rank} \mathcal{G}^{\infty}_{-\infty} = \widehat{m}_{R\infty}(-\infty,\infty) + \operatorname{rank} \mathcal{G}^{-\infty}_{\infty}, \tag{8.9}$$

$$\widehat{m}_{L-\infty}(-\infty,\infty) = \widehat{m}_{R\infty}(-\infty,\infty) \quad \text{if and only if} \quad \operatorname{rank} \mathcal{G}^{\infty}_{-\infty} = \operatorname{rank} \mathcal{G}^{-\infty}_{\infty}. \tag{8.10}$$

Condition (8.10) is a generalization of Proposition 1.3 to a possibly uncontrollable system (H). Indeed, if the system (H) is completely controllable on $\mathbb{R}$, then $\operatorname{rank} \mathcal{G}^{\infty}_{-\infty} = n = \operatorname{rank} \mathcal{G}^{-\infty}_{\infty}$ holds. Thus, we obtain from (8.10) the equality $\widehat{m}_{L-\infty}(-\infty,\infty) = \widehat{m}_{R\infty}(-\infty,\infty)$ saying that the principal solutions $\hat{Y}_{\infty}$ and $\hat{Y}_{-\infty}$ have the same number of focal points in $\mathbb{R}$, see also Example 7.3. Finally, from Remark 6.6 we obtain for any conjoined bases Y and $\tilde{Y}$ of (H) the equalities

$$m_L(-\infty,\infty) - \widetilde{m}_L(-\infty,\infty) = \mu_{\infty}(Y,\tilde{Y}) - \mu_{-\infty}(Y,\tilde{Y}), \tag{8.11}$$

$$m_R(-\infty,\infty) - \widetilde{m}_R(-\infty,\infty) = \mu^*_{-\infty}(Y,\tilde{Y}) - \mu^*_{\infty}(Y,\tilde{Y}). \tag{8.12}$$

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Appendix A. New results about antiprincipal solutions at infinity

In this section we present two new characterizations of antiprincipal solutions of (H) at ∞, which are related to their asymptotic behavior at ∞. The first result generalizes [27, Theorem 5.13] to the case when the involved principal and antiprincipal solutions at ∞ belong to two different genera of conjoined bases of (H). This result (or its Corollary A.3 presented below) is utilized in the proof of Theorem 3.3 about the multiplicities of focal points of conjoined bases at ∞.

Theorem A.1. *Assume that (1.1) holds with $\mathcal{I} = [a,\infty)$ and system (H) is nonoscillatory at ∞. Let $\hat{Y}$ be a principal solution of (H) at ∞ and Y be a conjoined basis of (H) and let $\alpha \in [a,\infty)$ be such that $d[\alpha,\infty) = d_{\infty}$ and $\hat{Y}$ and Y have constant kernel on the interval $[\alpha,\infty)$. Moreover, let $P_{\hat{\mathcal{S}}_{\alpha}\infty}$ and $P_{\mathcal{S}_{\alpha}\infty}$ be the associated orthogonal projectors in (2.33). Then Y is an antiprincipal solution of (H) at ∞ if and only if*

$$\operatorname{rank}\,[P_{\hat{\mathcal{S}}_{\alpha}\infty}\, W(\hat{Y}, Y)\, P_{\mathcal{S}_{\alpha}\infty}] = n - d_{\infty}. \tag{A.1}$$

Proof. Let $\hat{\mathcal{G}}^\infty$ and $\mathcal{G}^\infty$ be the genera of conjoined bases of (H), which correspond to $\hat{Y}$ and Y, respectively. Denote by $\mathcal{H}^\infty := \hat{\mathcal{G}}^\infty \wedge \mathcal{G}^\infty$ the infimum of $\hat{\mathcal{G}}^\infty$ and $\mathcal{G}^\infty$, see Subsection 2.5. Since $\mathcal{H}^\infty \preceq \hat{\mathcal{G}}^\infty$ and $\mathcal{H}^\infty \preceq \mathcal{G}^\infty$, we know from Proposition 2.8 that there exist conjoined bases $\hat{Y}_*$ and Y_* of (H) with constant kernel on $[\alpha, \infty)$, which are contained in $\hat{Y}$ and Y on $[\alpha, \infty)$, respectively, and which belong to the genus $\mathcal{H}^\infty$. Moreover, the conjoined basis $\hat{Y}_*$ is a principal solution of (H) at ∞, by Remark 2.10. According to [28, Lemma 5.2] we have the equality

$$P_{\hat{\mathcal{S}}_{\alpha\infty}}\, W(\hat{Y}_*, Y_*)\, P_{\mathcal{S}_{\alpha\infty}} = P_{\hat{\mathcal{S}}_{\alpha\infty}}\, W(\hat{Y}, Y)\, P_{\mathcal{S}_{\alpha\infty}}. \tag{A.2}$$

Now if Y is an antiprincipal solution of (H) at ∞, then also Y_* is an antiprincipal solution of (H) at ∞, by Remark 2.10. Consequently, the result in [23, Theorem 6.3.11] or [27, Theorem 5.13] implies the identity $\operatorname{rank}[P_{\hat{\mathcal{S}}_{\alpha\infty}}\, W(\hat{Y}_*, Y_*)\, P_{\mathcal{S}_{\alpha\infty}}] = n - d_\infty$. In particular, formula (A.1) then follows from (A.2). Conversely, if (A.1) holds, then $\operatorname{rank}[P_{\hat{\mathcal{S}}_{\alpha\infty}}\, W(\hat{Y}_*, Y_*)\, P_{\mathcal{S}_{\alpha\infty}}] = n - d_\infty$ by (A.2). Therefore, the result in [27, Theorem 5.13] yields that Y_* is an antiprincipal solution of (H) at ∞. Hence, Y is also an antiprincipal solution of (H) at ∞, by Remark 2.10. □

Remark A.2. (i) If $\hat{Y}$ in Theorem A.1 is the minimal principal solution at ∞, i.e., if $\hat{\mathcal{G}}^\infty = \mathcal{G}^\infty_{\min}$ in the above proof, then condition (A.1) reduces to $\operatorname{rank}[W(\hat{Y}, Y)\, P_{\mathcal{S}_{\alpha\infty}}] = n - d_\infty$.

(ii) Similarly, if Y in Theorem A.1 is a minimal conjoined basis of (H) near ∞, i.e., if $\mathcal{G}^\infty = \mathcal{G}^\infty_{\min}$ in the above proof, then condition (A.1) reduces to $\operatorname{rank}[P_{\hat{\mathcal{S}}_{\alpha\infty}} W(\hat{Y}, Y)] = n - d_\infty$.

(iii) In particular, if both $\hat{Y}$ and Y belong to the minimal genus $\mathcal{G}^\infty_{\min}$ near ∞, then condition (A.1) reduces to $\operatorname{rank} W(\hat{Y}, Y) = n - d_\infty$. This result is known in [27, Corollary 5.14].

The result in Theorem A.1 leads to a new interpretation of the limit characterization of principal solutions of (H) at ∞ in terms of antiprincipal solutions of (H) at ∞, resp. in terms of the Wronskian $W(\hat{Y}, Y)$. More precisely, we obtain the following simultaneous characterization of $\hat{Y}$ and Y being principal and antiprincipal solutions of (H) at ∞, which is in the light of Theorem A.1 an equivalent formulation of [28, Theorem 5.1].

Corollary A.3. *Assume that (1.1) holds with $\mathcal{I} = [a, \infty)$ and system (H) is nonoscillatory at ∞. Let $\hat{Y}$ and Y be two conjoined bases of (H) from given genera $\hat{\mathcal{G}}^\infty$ and $\mathcal{G}^\infty$, respectively. Let $\alpha \in [a, \infty)$ be such that $d[\alpha, \infty) = d_\infty$ and $\hat{Y}$ and Y have constant kernel on the interval $[\alpha, \infty)$. Denote by $P_{\hat{\mathcal{S}}_{\alpha\infty}}$ and $P_{\mathcal{S}_{\alpha\infty}}$ their associated orthogonal projectors in (2.33). Furthermore, let $\mathcal{H}^\infty = \hat{\mathcal{G}}^\infty \wedge \mathcal{G}^\infty$ be the infimum of the genera $\hat{\mathcal{G}}^\infty$ and $\mathcal{G}^\infty$ and let $\hat{P}_*$ be the orthogonal projector in Proposition 2.8 defined through $\hat{Y}$ and $\mathcal{H}^\infty$. Then the following statements are equivalent.*

(i) *The conjoined basis $\hat{Y}$ is a principal solution of (H) at ∞ and the conjoined basis Y is an antiprincipal solution of (H) at ∞.*

(ii) *The limit of $X^\dagger(t)\, \hat{X}(t)\, \hat{P}_*$ as $t \to \infty$ exists and satisfies*

$$\lim_{t\to\infty} X^\dagger(t)\, \hat{X}(t)\, \hat{P}_* = L \quad \textit{with} \quad \operatorname{Im} L^T = \operatorname{Im}(\hat{P}_* - P_{\hat{\mathcal{S}}_{\alpha\infty}}). \tag{A.3}$$

In this case the Wronskian $W(\hat{Y}, Y)$ satisfies condition (A.1).

Proof. Assume that condition (i) holds. Then Theorem A.1 yields that (A.1) is satisfied, and then condition (ii) follows from [28, Theorem 5.1]. Conversely, if condition (ii) holds, then we know by [28, Theorem 5.1] that $\hat{Y}$ is a principal solution of (H) at ∞ and (A.1) is satisfied. In turn, Theorem A.1 implies that Y is an antiprincipal solution of (H) at ∞, which shows that condition (i) holds. □

The result in Corollary A.3 poses a question about the existence of the limit in (A.3) when the conjoined basis $\hat{Y}$ is not in general a principal solution of (H) at ∞. We answer this question in the theorem below. At the same time we generalize [27, Theorem 6.3] to the case when the considered conjoined bases Y and $\tilde{Y}$ belong to two different genera $\mathcal{G}^\infty$ and $\tilde{\mathcal{G}}^\infty$, see the conjecture in [27, Remark 6.7(iv)]. Moreover, our new result also completes the statement in [27, Theorem 6.3] in a sense that it is actually an equivalence instead of an implication. The statement below (or its Corollary A.6) is utilized in the proof of Theorem 3.3.

Theorem A.4. *Assume that (1.1) holds with $\mathcal{I} = [a, \infty)$ and system (H) is nonoscillatory at ∞. Let Y and $\tilde{Y}$ be two conjoined bases of (H) belonging to given genera $\mathcal{G}^\infty$ and $\tilde{\mathcal{G}}^\infty$, respectively. Let $\alpha \in [a, \infty)$ be such that $d[\alpha, \infty) = d_\infty$ and Y and $\tilde{Y}$ have constant kernel on the interval $[\alpha, \infty)$. Moreover, let $\tilde{T}_{\alpha,\infty}$ and $P_{\tilde{\mathcal{S}}_\alpha\infty}$ be the matrices in (1.7) and (2.33), which correspond to $\tilde{Y}$. Denote by $\mathcal{H}^\infty$ the infimum of the genera $\tilde{\mathcal{G}}^\infty$ and $\mathcal{G}^\infty$ and let $\tilde{P}_*$ be the projector in Proposition 2.8 defined through $\tilde{Y}$ and $\mathcal{H}^\infty$. Then the following statements are equivalent.*

(i) *The conjoined basis Y is an antiprincipal solution of (H) at ∞.*
(ii) *The limit of $X^\dagger(t)\,\tilde{X}(t)\,\tilde{P}_*$ as $t \to \infty$ exists and satisfies*

$$\lim_{t\to\infty} X^\dagger(t)\,\tilde{X}(t)\,\tilde{P}_* = L \quad \textit{with} \quad \operatorname{Im} L^T = \operatorname{Im} \tilde{T}_{\alpha,\infty} \oplus \operatorname{Im}(\tilde{P}_* - P_{\tilde{\mathcal{S}}_\alpha\infty}). \tag{A.4}$$

Proof. Let P_* be the orthogonal projector in Proposition 2.8 defined through Y and $\mathcal{H}^\infty$ and let Y_* and $\tilde{Y}_*$ be the conjoined bases of (H) from the genus $\mathcal{H}^\infty$, which are contained in Y and $\tilde{Y}$ on $[\alpha, \infty)$ with respect to P_* and $\tilde{P}_*$, respectively. Moreover, let $R_*(t)$ and $\tilde{R}_*(t)$ be the matrices in (2.31) associated with Y_* and $\tilde{Y}_*$. By using the properties of genera of conjoined bases and the relation "being contained" and (2.41) we have the equalities

$$\tilde{R}_*(t) = R_*(t), \quad \tilde{X}_*(t) = \tilde{X}(t)\,\tilde{P}_*, \quad X_*^\dagger(t) = X^\dagger(t)\,R_*(t), \quad t \in [\alpha, \infty). \tag{A.5}$$

In particular, the identities in (A.5) together with $\tilde{R}_*(t)\,\tilde{X}_*(t) = \tilde{X}_*(t)$ on $[\alpha, \infty)$ yield

$$\begin{aligned} X^\dagger(t)\,\tilde{X}(t)\,\tilde{P}_* &\overset{(A.5)}{=} X^\dagger(t)\,\tilde{X}_*(t) = X^\dagger(t)\,\tilde{R}_*(t)\,\tilde{X}_*(t) \\ &\overset{(A.5)}{=} X^\dagger(t)\,R_*(t)\,\tilde{X}_*(t) \overset{(A.5)}{=} X_*^\dagger(t)\,\tilde{X}_*(t), \quad t \in [\alpha, \infty). \end{aligned} \tag{A.6}$$

In this way we transfer the study of the limit in (A.4) into the genus $\mathcal{H}^\infty$, where we can apply the known result from [27, Theorem 6.3]. Let $S_\alpha(t)$, $\tilde{S}_\alpha(t)$, $S_{*\alpha}(t)$, $\tilde{S}_{*\alpha}(t)$, and (suppressing the index ∞) $T_\alpha := T_{\alpha,\infty}$, $\tilde{T}_\alpha := \tilde{T}_{\alpha,\infty}$, $T_{*\alpha} := T_{*\alpha,\infty}$, $\tilde{T}_{*\alpha} := \tilde{T}_{*\alpha,\infty}$ be the corresponding S-matrices and T-matrices defined in (1.7). According to Remark 2.7(i) we have the equalities

$$S_\alpha(t) = S_{*\alpha}(t), \quad \tilde{S}_\alpha(t) = \tilde{S}_{*\alpha}(t), \quad t \in [\alpha, \infty), \tag{A.7}$$

$$P_{\mathcal{S}_\alpha\infty} = P_{\mathcal{S}_{*\alpha}\infty}, \quad P_{\tilde{\mathcal{S}}_\alpha\infty} = P_{\tilde{\mathcal{S}}_{*\alpha}\infty}, \quad T_\alpha = T_{*\alpha}, \quad \tilde{T}_\alpha = \tilde{T}_{*\alpha}, \tag{A.8}$$

where $P_{\mathcal{S}_\alpha\infty}$, $P_{\mathcal{S}_{*\alpha}\infty}$, and $P_{\tilde{\mathcal{S}}_{*\alpha}\infty}$ are the orthogonal projectors in (2.33) associated with Y, Y_*, and $\tilde{Y}_*$, respectively. Moreover, based on the first equality in (A.5) the conjoined bases Y_* and $\tilde{Y}_*$ are mutually representable on $[\alpha, \infty)$ in the spirit of Proposition 2.5 with $Y_1 := \tilde{Y}_*$ and $Y_2 := Y_*$. More precisely, there exist constant matrices $\tilde{M}_*, \tilde{N}_* \in \mathbb{R}^{n\times n}$ such that $\tilde{M}_*$ is nonsingular, $\tilde{M}_*^T \tilde{N}_*$ is symmetric, and the identities

$$X_*(t) = \tilde{X}_*(t)\,[\tilde{P}_* \tilde{M}_* + \tilde{S}_{*\alpha}(t)\,\tilde{N}_*], \quad \tilde{X}_*(t) = X_*(t)\,[P_* \tilde{M}_*^{-1} - S_{*\alpha}(t)\,\tilde{N}_*^T] \tag{A.9}$$

hold on $[\alpha, \infty)$, by (2.39). In particular, by using the second formula in (A.9) and the equalities $X_*^\dagger(t)\,X_*(t) = P_*$ and $P_*\,S_{*\alpha}(t) = S_{*\alpha}(t)$ on $[\alpha, \infty)$ we obtain that

$$X_*^\dagger(t)\,\tilde{X}_*(t) = X_*^\dagger(t)\,X_*(t)\,[P_* \tilde{M}_*^{-1} - S_{*\alpha}(t)\,\tilde{N}_*^T] = P_* \tilde{M}_*^{-1} - S_{*\alpha}(t)\,\tilde{N}_*^T, \quad t \in [\alpha, \infty). \tag{A.10}$$

Next, let Y_{**} and $\tilde{Y}_{**}$ be the conjoined bases of (H), which are contained in Y_* and $\tilde{Y}_*$ on $[\alpha, \infty)$ with respect to the orthogonal projectors $P_{\mathcal{S}_{*\alpha}\infty}$ and $P_{\tilde{\mathcal{S}}_{*\alpha}\infty}$, respectively. Since both Y_{**} and $\tilde{Y}_{**}$ have constant kernel on $[\alpha, \infty)$ and by Remark 2.7(ii) they belong to the (unique) minimal genus $\mathcal{G}_{\min}^\infty$, it then follows that Y_{**} and $\tilde{Y}_{**}$ are mutually representable on $[\alpha, \infty)$. In particular, if in agreement with (2.37) in Proposition 2.5 with $Y_1 := \tilde{Y}_{**}$ and $Y_2 := Y_{**}$ the conjoined basis Y_{**} is expressed in terms of the conjoined basis $\tilde{Y}_{**}$ via matrices $\tilde{M}_{**}$ and $\tilde{N}_{**}$, then $\tilde{M}_{**}$ is nonsingular, $\tilde{M}_{**}^T \tilde{N}_{**}$ is symmetric, and according to [26, Lemma 6.9] we have the formulas

$$\tilde{P}_* \tilde{M}_* P_{\mathcal{S}_{*\alpha}\infty} = P_{\tilde{\mathcal{S}}_{*\alpha}\infty} \tilde{M}_{**}, \quad P_* \tilde{M}_*^{-1} P_{\tilde{\mathcal{S}}_{*\alpha}\infty} = P_{\mathcal{S}_{*\alpha}\infty} \tilde{M}_{**}^{-1}, \quad \tilde{N}_{**} = P_{\tilde{\mathcal{S}}_{*\alpha}\infty} \tilde{N}_* P_{\mathcal{S}_{*\alpha}\infty}. \tag{A.11}$$

Moreover, denoting by $T_{**\alpha} := T_{**\alpha,\infty}$ and $\tilde{T}_{**\alpha} := \tilde{T}_{**\alpha,\infty}$ the corresponding T-matrices defined in (1.7), we obtain by Remark 2.7(i) that $T_{**\alpha} = T_{*\alpha}$ and $\tilde{T}_{**\alpha} = \tilde{T}_{*\alpha}$. Hence, by (2.42) in Remark 2.7(ii) we get

$$T_{*\alpha} = T_{**\alpha} \overset{(2.42)}{=} \tilde{M}_{**}^T \tilde{T}_{**\alpha} \tilde{M}_{**} + \tilde{M}_{**}^T \tilde{N}_{**} = \tilde{M}_{**}^T \tilde{T}_{*\alpha} \tilde{M}_{**} + \tilde{M}_{**}^T \tilde{N}_{**}. \tag{A.12}$$

Suppose now that statement (i) holds, i.e., Y is an antiprincipal solution of (H) at ∞. Then also Y_* is an antiprincipal solution at ∞ by Remark 2.10, and according to [27, Theorem 6.3] (with $\mathcal{G} := \mathcal{H}^\infty$, $(X, U) := Y_*$, and $(\tilde{X}, \tilde{U}) := \tilde{Y}_*$) we have that

$$\lim_{t\to\infty} X_*^\dagger(t)\,\tilde{X}_*(t) = L \quad \text{with} \quad \operatorname{Im} L^T = \operatorname{Im} \tilde{T}_{*\alpha} \oplus \operatorname{Im}(\tilde{P}_* - P_{\tilde{\mathcal{S}}_{*\alpha}\infty}). \tag{A.13}$$

In turn, with the aid of (A.6) and (A.8) the relation in (A.13) is equivalent with (A.4). This shows the validity of statement (ii).

Conversely, assume (ii), i.e., (A.4) holds. Denote by $L_0 := P_* \tilde{M}_*^{-1} - L$, where L is given in (A.4). Then by (A.6) and (A.8) we have that Y_* and $\tilde{Y}_*$ satisfy (A.13). Consequently, by (A.10) we get $S_{*\alpha}(t)\,\tilde{N}_*^T \to L_0$ as $t \to \infty$. Moreover, the definitions of the matrices $P_{\mathcal{S}_{*\alpha}\infty}$ and $T_{*\alpha}$ in (2.33) and (1.7) yield

$$P_{\mathcal{S}_{*\alpha}\infty}\,\tilde{N}_*^T = \lim_{t\to\infty} S_{*\alpha}^{\dagger}(t)\,S_{*\alpha}(t)\,\tilde{N}_*^T = \lim_{t\to\infty} S_{*\alpha}^{\dagger}(t) \times \lim_{t\to\infty} S_{*\alpha}(t)\,\tilde{N}_*^T = T_{*\alpha}\,L_0. \tag{A.14}$$

This implies that $\operatorname{Ker} L_0 \subseteq \operatorname{Ker}(P_{\mathcal{S}_{*\alpha}\infty}\,\tilde{N}_*^T)$. On the other hand, for any $v \in \operatorname{Ker}(P_{\mathcal{S}_{*\alpha}\infty}\,\tilde{N}_*^T)$ we have that $v \in \operatorname{Ker}[S_{*\alpha}(t)\,\tilde{N}_*^T]$ for all sufficiently large $t \in [\alpha,\infty)$ and hence, $v \in \operatorname{Ker} L_0$. Therefore, $\operatorname{Ker} L_0 = \operatorname{Ker}(P_{\mathcal{S}_{*\alpha}\infty}\,\tilde{N}_*^T)$ holds, which is equivalent with $\operatorname{Im} L_0^T = \operatorname{Im}(\tilde{N}_*\,P_{\mathcal{S}_{*\alpha}\infty})$. By combining the latter identity with the last formula in (A.11) we get

$$\operatorname{Im}(P_{\tilde{\mathcal{S}}_{*\alpha}\infty}\,L_0^T) = \operatorname{Im}(P_{\tilde{\mathcal{S}}_{*\alpha}\infty}\,\tilde{N}_*\,P_{\mathcal{S}_{*\alpha}\infty}) \overset{\text{(A.11)}}{=} \operatorname{Im}\tilde{N}_{**}. \tag{A.15}$$

In addition, from the definition of L_0 and the second identity in (A.11) it follows that

$$L_0\,P_{\tilde{\mathcal{S}}_{*\alpha}\infty} = P_*\,\tilde{M}_*^{-1}\,P_{\tilde{\mathcal{S}}_{*\alpha}\infty} - L\,P_{\tilde{\mathcal{S}}_{*\alpha}\infty} \overset{\text{(A.11)}}{=} P_{\mathcal{S}_{*\alpha}\infty}\,\tilde{M}_{**}^{-1} - L\,P_{\tilde{\mathcal{S}}_{*\alpha}\infty}. \tag{A.16}$$

Furthermore, since by (A.13) the inclusion $\operatorname{Im} L^T \subseteq \operatorname{Im}\tilde{T}_{*\alpha} \oplus \operatorname{Im}(\tilde{P}_* - P_{\tilde{\mathcal{S}}_{*\alpha}\infty})$ holds, utilizing the equalities $P_{\tilde{\mathcal{S}}_{*\alpha}\infty}\tilde{T}_{*\alpha} = \tilde{T}_{*\alpha}$ and $P_{\tilde{\mathcal{S}}_{*\alpha}\infty}(\tilde{P}_* - P_{\tilde{\mathcal{S}}_{*\alpha}\infty}) = 0$ we obtain that

$$\operatorname{Im}(P_{\tilde{\mathcal{S}}_{*\alpha}\infty}\,L^T) \subseteq \operatorname{Im}\tilde{T}_{*\alpha}, \quad \text{or equivalently} \quad \operatorname{Ker}\tilde{T}_{*\alpha} \subseteq \operatorname{Ker}(L\,P_{\tilde{\mathcal{S}}_{*\alpha}\infty}). \tag{A.17}$$

Next, we will prove the inclusion $\operatorname{Ker} T_{*\alpha} \subseteq \operatorname{Ker} P_{\mathcal{S}_{*\alpha}\infty}$. Let $v \in \operatorname{Ker} T_{*\alpha}$. The last identity in (A.11), the symmetry of $P_{\mathcal{S}_{*\alpha}\infty}$ and $T_{*\alpha}$, and equality (A.14) then yield

$$\tilde{N}_{**}\,v \overset{\text{(A.11)}}{=} P_{\tilde{\mathcal{S}}_{*\alpha}\infty}\,\tilde{N}_*\,P_{\mathcal{S}_{*\alpha}\infty}\,v \overset{\text{(A.14)}}{=} P_{\tilde{\mathcal{S}}_{*\alpha}\infty}\,L_0^T\,T_{*\alpha}\,v = 0. \tag{A.18}$$

Consequently, by using (A.12) and (A.18) we get $\tilde{M}_{**}^T\,\tilde{T}_{*\alpha}\,\tilde{M}_{**}\,v = T_{*\alpha}\,v - \tilde{M}_{**}^T\,\tilde{N}_{**}\,v = 0$. And since the matrix $\tilde{M}_{**}$ is nonsingular, we have $\tilde{M}_{**}\,v \in \operatorname{Ker}\tilde{T}_{*\alpha}$ and hence, $\tilde{M}_{**}\,v \in \operatorname{Ker}(L\,P_{\tilde{\mathcal{S}}_{*\alpha}\infty})$, by (A.17). On the other hand, formula (A.15) and the symmetry of $\tilde{M}_{**}^T\,\tilde{N}_{**}$ imply that

$$\operatorname{Im}(\tilde{M}_{**}^T\,P_{\tilde{\mathcal{S}}_{*\alpha}\infty}\,L_0^T) = \operatorname{Im}(\tilde{N}_{**}^T\,\tilde{M}_{**}), \quad \text{i.e.,} \quad \operatorname{Ker}(L_0\,P_{\tilde{\mathcal{S}}_{*\alpha}\infty}\,\tilde{M}_{**}) = \operatorname{Ker}(\tilde{M}_{**}^T\,\tilde{N}_{**}).$$

And since the equality $\tilde{N}_{**}\,v = 0$ holds, we have $\tilde{M}_{**}\,v \in \operatorname{Ker}(L_0\,P_{\tilde{\mathcal{S}}_{*\alpha}\infty})$. Thus, the vector $\tilde{M}_{**}\,v \in \operatorname{Ker}(L\,P_{\tilde{\mathcal{S}}_{*\alpha}\infty}) \cap \operatorname{Ker}(L_0\,P_{\tilde{\mathcal{S}}_{*\alpha}\infty})$. Moreover, from (A.16) it then follows that

$$P_{\mathcal{S}_{*\alpha}\infty}\,v = P_{\mathcal{S}_{*\alpha}\infty}\,\tilde{M}_{**}^{-1}\,\tilde{M}_{**}\,v \overset{\text{(A.16)}}{=} (L_0\,P_{\tilde{\mathcal{S}}_{*\alpha}\infty} + L\,P_{\tilde{\mathcal{S}}_{*\alpha}\infty})\,\tilde{M}_{**}\,v = 0.$$

Therefore, $v \in P_{\mathcal{S}_{*\alpha}\infty}$ and the inclusion $\operatorname{Ker} T_{*\alpha} \subseteq \operatorname{Ker} P_{\mathcal{S}_{*\alpha}\infty}$ is established. This means that $\operatorname{Im} P_{\mathcal{S}_{*\alpha}\infty} \subseteq \operatorname{Im} T_{*\alpha}$. On the other hand, the opposite inclusion $\operatorname{Im} T_{*\alpha} \subseteq \operatorname{Im} P_{\mathcal{S}_{*\alpha}\infty}$ always holds by (2.34) and hence, we have $\operatorname{Im} T_{*\alpha} = \operatorname{Im} P_{\mathcal{S}_{*\alpha}\infty}$ and $\operatorname{rank} T_{*\alpha} = n - d[\alpha,\infty)$. Therefore, the conjoined basis Y_* is an antiprincipal solution of (H) at ∞. In turn, by Remark 2.10 the conjoined basis Y is also an antiprincipal solution of (H) at ∞. The proof is complete. □

Remark A.5. The proof of Theorem A.4 implies that the second condition in (A.4) can be equivalently replaced by the inclusion

$$\operatorname{Im} L^T \subseteq \operatorname{Im} \tilde{T}_{\alpha,\infty} \oplus \operatorname{Im}(\tilde{P}_* - P_{\tilde{\mathcal{S}}_\alpha\infty}). \tag{A.19}$$

Indeed, in the proof of (i) ⇒ (ii) we showed that (A.13) holds, which by $\tilde{T}_\alpha = \tilde{T}_{*\alpha}$ from (A.8) implies (A.19). In the opposite direction (ii) ⇒ (i) we utilized just the inclusion (A.19).

In the last part of this section we present several special cases of Theorem A.4, which are also new in a sense that they extend known statements in the literature. These special cases are concerned with the situations when either $\mathcal{G}^\infty$ is the maximal genus $\mathcal{G}^\infty_{\max}$, or one of the genera $\mathcal{H}^\infty$ of $\tilde{\mathcal{G}}^\infty$ is the minimal genus $\mathcal{G}^\infty_{\min}$. These considerations will also show that Theorem A.4 is new even for the completely controllable system (H).

First we consider the situation when Y is a maximal conjoined basis of (H) near ∞. In this case we obtain from Theorem A.4 the following characterization of maximal antiprincipal solutions of (H) at ∞.

Corollary A.6. *Assume that (1.1) holds with $\mathcal{I} = [a,\infty)$ and system (H) is nonoscillatory at ∞. Let Y be a maximal conjoined basis of (H) near ∞ and let $\tilde{Y}$ be a conjoined basis of (H). Let $\alpha \in [a,\infty)$ be such that $d[\alpha,\infty) = d_\infty$, the matrix $X(t)$ is invertible on $[\alpha,\infty)$, and $\tilde{Y}$ has constant kernel on $[\alpha,\infty)$. Moreover, let $\tilde{T}_{\alpha,\infty}$, $\tilde{P}$, and $P_{\tilde{\mathcal{S}}_\alpha\infty}$ be the matrices in (1.7), (2.31), and (2.33), which correspond to $\tilde{Y}$. Then the following statements are equivalent.*

(i) *The conjoined basis Y is a maximal antiprincipal solution of (H) at ∞.*
(ii) *The limit of $X^{-1}(t)\,\tilde{X}(t)$ as $t \to \infty$ exists and satisfies*

$$\lim_{t\to\infty} X^{-1}(t)\,\tilde{X}(t) = L \quad \textit{with} \quad \operatorname{Im} L^T = \operatorname{Im} \tilde{T}_{\alpha,\infty} \oplus \operatorname{Im}(\tilde{P} - P_{\tilde{\mathcal{S}}_\alpha\infty}). \tag{A.20}$$

Proof. The result follows from Theorem A.4 for the choice of $\mathcal{G}^\infty := \mathcal{G}^\infty_{\max}$. In this case we have $\mathcal{H}^\infty = \mathcal{G}^\infty_{\max} \wedge \tilde{\mathcal{G}}^\infty = \tilde{\mathcal{G}}^\infty$, and hence $\tilde{P}_* = \tilde{P}$ and $\tilde{X}(t)\,\tilde{P}_* = \tilde{X}(t)\,\tilde{P} = \tilde{X}(t)$ on $[\alpha,\infty)$. □

The other special cases of Theorem A.4 are concerned with the situation when $\mathcal{H}^\infty$ is the minimal genus $\mathcal{G}^\infty_{\min}$. The notation and context of the following remark refers to Theorem A.4.

Remark A.7. (i) If $\mathcal{H}^\infty = \mathcal{G}^\infty \wedge \tilde{\mathcal{G}}^\infty = \mathcal{G}^\infty_{\min}$, then $\tilde{P}_* = P_{\tilde{\mathcal{S}}_\alpha\infty}$ and condition (A.4) reduces to

$$\lim_{t\to\infty} X^\dagger(t)\,\tilde{X}(t)\,P_{\tilde{\mathcal{S}}_\alpha\infty} = L \quad \text{with} \quad \operatorname{Im} L^T = \operatorname{Im} \tilde{T}_{\alpha,\infty}. \tag{A.21}$$

In this case Theorem A.4 with (A.21) extends [28, Corollary 5.3] to the situation when $\tilde{Y}$ is not necessarily a principal solution of (H) at ∞.

(ii) If $\tilde{\mathcal{G}}^\infty := \mathcal{G}^\infty_{\min}$, then part (i) of this remark applies with $\tilde{P} = P_{\tilde{\mathcal{S}}_\alpha\infty}$ and (A.21) reduces to

$$\lim_{t\to\infty} X^\dagger(t)\,\tilde{X}(t) = L \quad \text{with} \quad \operatorname{Im} L^T = \operatorname{Im} \tilde{T}_{\alpha,\infty}. \tag{A.22}$$

In this case Theorem A.4 with (A.22) extends [28, Corollary 5.4] to the situation when $\tilde{Y}$ is not necessarily a principal solution of (H) at ∞.

(iii) If $\mathcal{G}^\infty := \mathcal{G}^\infty_{\max}$ and $\tilde{\mathcal{G}}^\infty := \mathcal{G}^\infty_{\min}$, then part (ii) of this remark holds and (A.22) reduces to

$$\lim_{t\to\infty} X^{-1}(t)\, \tilde{X}(t) = L \quad \text{with} \quad \operatorname{Im} L^T = \operatorname{Im} \tilde{T}_{\alpha,\infty}. \tag{A.23}$$

In this case Theorem A.4 with (A.23) extends [28, Corollary 5.5] to the situation when $\tilde{Y}$ is not necessarily a principal solution of (H) at ∞. Note that condition (A.23) is also a special case of (A.20) in Corollary A.6 for the choice of $\tilde{\mathcal{G}}^\infty := \mathcal{G}^\infty_{\min}$.

Finally, we comment separately the situation of a completely controllable system (H). Even in this very special case the result in Theorem A.4 (or Remark A.7(iii) with $\mathcal{G}^\infty_{\max} = \mathcal{G}^\infty_{\min}$) is new and it reads as follows. We also note that the assumption of the eventual complete controllability of (H) can be replaced by the weaker condition $d_\infty = 0$ with the same conclusion. The latter condition is a version of a weak controllability condition used in [15, Hypothesis 2.7] or in [17, Condition D2$_\omega$, p. 260].

Corollary A.8. *Assume that* (1.1) *holds with $\mathcal{I} = [a,\infty)$ and system* (H) *is nonoscillatory at ∞ and eventually completely controllable. Let Y and $\tilde{Y}$ be two conjoined basis of* (H). *Let $\alpha \in [a,\infty)$ be such that the matrices $X(t)$ and $\tilde{X}(t)$ are invertible on $[\alpha,\infty)$ and let $\tilde{T}_{\alpha,\infty}$ be the matrix in* (1.7)*, which corresponds to $\tilde{Y}$. Then the following statements are equivalent.*

(i) *The conjoined basis Y is an antiprincipal solution of* (H) *at ∞.*
(ii) *The limit of $X^{-1}(t)\, \tilde{X}(t)$ as $t \to \infty$ exists and satisfies* (A.23).

Remark A.9. Following the last parts of Subsections 2.5 and 2.7 we conclude that the results in this section hold without any change for genera of conjoined bases of a nonoscillatory system (H) at $-\infty$, with the orthogonal projectors and limits considered for $t \to -\infty$.

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APPENDIX E

Paper by Šepitka & Šimon Hilscher (JDE 2020)

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Singular Sturmian comparison theorems for linear Hamiltonian systems ☆

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Abstract

In this paper we prove singular comparison theorems on unbounded intervals for two nonoscillatory linear Hamiltonian systems satisfying the Sturmian majorant condition and the Legendre condition. At the same time we do not impose any controllability condition. The results are phrased in terms of the comparative index and the numbers of proper focal points of the (minimal) principal solutions of these systems at both endpoints of the considered interval. The main idea is based on an application of new transformation theorems for principal and antiprincipal solutions at infinity and on new limit properties of the comparative index involving these solutions. This work generalizes the recently obtained Sturmian separation theorems on unbounded intervals for one system by the authors (2019), as well as the Sturmian comparison theorems and transformation theorems on compact intervals by J. Elyseeva (2016 and 2018). We note that all the results are new even in the completely controllable case.

MSC: 34C10

Keywords: Sturmian comparison theorem; Linear Hamiltonian system; Proper focal point; Minimal principal solution; Antiprincipal solution; Comparative index

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1. Introduction

We consider the linear Hamiltonian differential systems

$$y' = \mathcal{J}\mathcal{H}(t)\, y, \quad t \in \mathcal{I}, \tag{H}$$

$$\hat{y}' = \mathcal{J}\hat{\mathcal{H}}(t)\, \hat{y}, \quad t \in \mathcal{I}, \tag{$\hat{\text{H}}$}$$

where $\mathcal{I} \subseteq \mathbb{R}$ is a fixed interval and $\mathcal{H}, \hat{\mathcal{H}} : \mathcal{I} \to \mathbb{R}^{2n\times 2n}$ are given piecewise continuous symmetric matrix-valued functions on $\mathcal{I}$ satisfying the Sturmian majorant condition

$$\mathcal{H}(t) \geq \hat{\mathcal{H}}(t) \quad \text{for all } t \in \mathcal{I}. \tag{1.1}$$

In this setting we say that system (H) is a Sturmian majorant of ($\hat{\text{H}}$), or that system ($\hat{\text{H}}$) is a Sturmian minorant of (H). We assume that $n \in \mathbb{N}$ is a given dimension and $\mathcal{J} \in \mathbb{R}^{2n\times 2n}$ is the canonical skew-symmetric matrix (see equation (2.12) below). In addition to (1.1) we assume that the minorant system ($\hat{\text{H}}$) satisfies the Legendre condition

$$\hat{B}(t) \geq 0 \quad \text{for all } t \in \mathcal{I}, \tag{1.2}$$

where $\hat{B}(t)$ is the lower right $n \times n$ block of $\hat{\mathcal{H}}(t)$. Along with the basic systems (H) and ($\hat{\text{H}}$) we will also consider a certain transformed linear Hamiltonian system

$$\tilde{y}' = \mathcal{J}\tilde{\mathcal{H}}(t)\, \tilde{y}, \quad t \in \mathcal{I}, \tag{$\tilde{\text{H}}$}$$

which is related to (H) and ($\hat{\text{H}}$) by a symplectic transformation (see Remark 1.4 below).

We are interested in the Sturmian comparison theorems, which provide a way for the estimation of the number of focal points of conjoined bases of the majorant system (H) in terms of the number of focal points of conjoined bases of the minorant system ($\hat{\text{H}}$), or vice versa. The novel approach of this paper resides in four aspects: (i) we consider an unbounded interval $\mathcal{I}$ and thus derive the singular Sturmian comparison theorems, (ii) we remove the controllability assumption on the involved systems, (iii) we obtain exact formulas for the numbers of focal points of conjoined bases of these two systems, and (iv) as key tools we derive new results in the transformation theory of principal and antiprincipal solutions at ∞ and new limit properties of the comparative index involving these solutions.

Solutions of systems (H) or ($\hat{\text{H}}$) are piecewise continuously differentiable functions on $\mathcal{I}$. We will consider $2n$-vector-valued solutions denoted by small letters (typically y or $\hat{y}$) or $2n \times n$-matrix-valued solutions denoted by capital letters (typically Y or $\hat{Y}$). We will split the vector solutions into their n-vector components $y = (x^T, u^T)^T$ or the matrix solutions into their $n \times n$-matrix components $Y = (X^T, U^T)^T$, see notation (2.14) below. In the theory of uncontrollable linear Hamiltonian systems it is known that the conjoined bases may have the X-component singular on a nondegenerate subinterval of $\mathcal{I}$. More precisely, the result of [23, Theorem 3] or [18, Proof of Lemma 3.6(a)] shows that under the Legendre condition the kernel of $X(t)$ is piecewise constant on $\mathcal{I}$ for any conjoined basis Y. This means that the kernel of $X(t)$ changes finitely many times in any compact subinterval of $\mathcal{I}$ and we say that Y has a *left proper focal point* at $t_0 \in \mathcal{I}$ if $\operatorname{Ker} X(t_0^-) \subsetneqq \operatorname{Ker} X(t_0)$ with the multiplicity

$$m_L(t_0) := \operatorname{def} X(t_0) - \operatorname{def} X(t_0^-), \tag{1.3}$$

and a *right proper focal point* at $t_0 \in \mathcal{I}$ if $\operatorname{Ker} X(t_0^+) \subsetneqq \operatorname{Ker} X(t_0)$ with the multiplicity

$$m_R(t_0) := \operatorname{def} X(t_0) - \operatorname{def} X(t_0^+). \tag{1.4}$$

These notions were defined in [24,36]. For brevity, the adjective "proper" will be disregarded in the subsequent terminology. The notations $\operatorname{Ker} X(t_0^\pm)$, $\operatorname{def} X(t_0^\pm)$, and later $\operatorname{rank} X(t_0^\pm)$ represent the one-sided limits at t_0 of the piecewise constant quantities $\operatorname{Ker} X(t)$, $\operatorname{def} X(t) := \dim \operatorname{Ker} X(t)$, and $\operatorname{rank} X(t)$. When counting the left and right focal points of conjoined bases Y of (H) in the interval $\mathcal{I}$ we will use the notation $m_L(\mathcal{I})$ and $m_R(\mathcal{I})$, that is,

$$m_L(\mathcal{I}) := \sum_{t_0 \in \mathcal{I}} m_L(t_0), \quad m_R(\mathcal{I}) := \sum_{t_0 \in \mathcal{I}} m_R(t_0). \tag{1.5}$$

In a similar way we will use the notation $\widehat{m}_L(\mathcal{I})$ and $\widehat{m}_R(\mathcal{I})$ for a conjoined basis $\hat{Y}$ of ($\hat{\text{H}}$). For special conjoined bases Y_{t_0} and $\hat{Y}_{t_0}$, called the *principal solutions* of (H) and ($\hat{\text{H}}$) at $t_0 \in \mathcal{I}$ and defined by the initial conditions

$$Y_{t_0}(t_0) = E, \quad \hat{Y}_{t_0}(t_0) = E, \quad E := (0,\ I)^T, \tag{1.6}$$

we will use the notation $m_{Lt_0}(\mathcal{I})$, $m_{Rt_0}(\mathcal{I})$, $\widehat{m}_{Lt_0}(\mathcal{I})$, $\widehat{m}_{Rt_0}(\mathcal{I})$. The brackets around $\mathcal{I}$ will be dropped when considering an interval $\mathcal{I}$ with specific endpoints. The focal points are always counted including their multiplicities.

Oscillation theory of linear Hamiltonian systems represents a classical topic in the qualitative theory of differential equations. Standard references include the monographs [4,9,19,22,25,26] or more recently [2,21,27]. Regarding the compact interval $\mathcal{I} = [a, b]$, the classical Sturmian comparison theorem for the second order Sturm–Liouville differential equations is presented in [19, Theorem XI.3.1] or [26, Theorem II.3.2(a)]. An extension of this result to controllable linear Hamiltonian systems (H) and ($\hat{\text{H}}$) was derived in [8, Theorem 4] by Coppel, in [3, pg. 252] by Arnold (also quoted in [27, Theorem 4.8]), and in [22, Section 7.3] by Kratz. In particular, the results in [22, Corollary 7.3.2] use the principal solutions of (H) and ($\hat{\text{H}}$) at the endpoints a and b as the reference solutions for counting the focal points. We recall that system (H) is called *completely controllable* (or *identically normal*) on the interval $\mathcal{I}$ if the only solution $y = (x^T(\cdot) \equiv 0, u^T)^T$ of (H) on a nondegenerate subinterval $\mathcal{I}_0 \subseteq \mathcal{I}$ is the trivial solution $y(\cdot) \equiv 0$. Note that in this case the quantities in (1.3) and (1.4) coincide with the usual multiplicity of a focal point at t_0, which is defined by (see [22, Theorem 3.1.2])

$$m(t_0) := \operatorname{def} X(t_0) = \dim \operatorname{Ker} X(t_0). \tag{1.7}$$

Regarding an open or unbounded interval $\mathcal{I}$, a singular Sturmian comparison theorem for the second order Sturm–Liouville differential equations was obtained in [1, Theorem 1(i)] by Aharonov and Elias. Moreover, the following singular comparison theorem for controllable systems (H) and ($\hat{\text{H}}$) on $\mathcal{I} = [a, \infty)$ was derived in [12, Theorem 2] by Došlý and Kratz. The authors of [1,12] replaced the principal solution at b by the principal solution at ∞ in their comparison theorems. We recall, see e.g. [10], that the principal solution of (H) at ∞ is defined as its conjoined basis Y_∞ with $X_\infty(t)$ invertible on $[\alpha, \infty)$ for some $\alpha \in [a, \infty)$ and

$$\lim_{t\to\infty}\left(\int_{\alpha}^{t} X_{\infty}^{-1}(s)\, B(s)\, X_{\infty}^{T-1}(s)\, \mathrm{d}s\right)^{-1} = 0. \tag{1.8}$$

Proposition 1.1. *Assume that (1.1) and (1.2) hold on $\mathcal{I} = [a, \infty)$ and that systems (H) and ($\hat{\mathrm{H}}$) are nonoscillatory and completely controllable. Then the number of focal points of any conjoined basis Y of (H) in the interval $[a, \infty)$ is bounded from below by the number of focal points of the principal solution of ($\hat{\mathrm{H}}$) at ∞ in this interval. This means, in the above notation,*

$$m[a, \infty) \geq \widehat{m}_{\infty}[a, \infty). \tag{1.9}$$

The first Sturmian comparison theorems for uncontrollable linear Hamiltonian systems were derived in [34, Theorems 1.2 and 1.3] by the second author for a compact interval $\mathcal{I} = [a, b]$ and for the left focal points. These results are translated easily via [24, Remark 4.7] to the right focal points. The comparison theorems in [22, Section 7.3] and [34] were derived by using the oscillation theorems for self-adjoint eigenvalue problems and they are formulated as estimates or inequalities (unless $\hat{\mathcal{H}}(t) \equiv \mathcal{H}(t)$ on $\mathcal{I}$). On the other hand, the following exact formula for expressing the numbers of left and right focal points of conjoined bases Y and $\hat{Y}$ of (H) and ($\hat{\mathrm{H}}$) on $\mathcal{I} = [a, b]$ was derived in [15, Theorem 2.2]. This result is based on using the comparative index $\mu(\cdot, \cdot)$ and the dual comparative index $\mu^*(\cdot, \cdot)$ of Elyseeva [13,14], which we define in Section 2.

Proposition 1.2. *Assume that (1.1) and (1.2) hold on $\mathcal{I} = [a, b]$ and let Y and $\hat{Y}$ be any conjoined bases of (H) and ($\hat{\mathrm{H}}$). Let $\hat{Z}$ be a fundamental matrix of ($\hat{\mathrm{H}}$) satisfying $\hat{Y}(t) = \hat{Z}(t)\, E$ on $[a, b]$, where the matrix E is given in (1.6), and consider the function $\tilde{Y}(t) := \hat{Z}^{-1}(t)\, Y(t)$ on $[a, b]$. Then the comparative index $\mu(Y(t), \hat{Y}(t))$ is piecewise constant and right-continuous on $[a, b]$ and for every $t_0 \in (a, b]$ the multiplicities $m_L(t_0)$, $\widehat{m}_L(t_0)$, and $\widetilde{m}_L(t_0)$ of left focal points of Y, $\hat{Y}$, and $\tilde{Y}$ at t_0 defined through (1.3) satisfy the equality*

$$m_L(t_0) - \widehat{m}_L(t_0) = \widetilde{m}_L(t_0) + \mu(Y(t_0), \hat{Y}(t_0)) - \lim_{t\to t_0^-} \mu(Y(t), \hat{Y}(t)). \tag{1.10}$$

Moreover, the numbers of left focal points of Y and $\hat{Y}$ in $(a, b]$ are connected by

$$m_L(a, b] - \widehat{m}_L(a, b] = \widetilde{m}_L(a, b] + \mu(Y(b), \hat{Y}(b)) - \mu(Y(a), \hat{Y}(a)), \tag{1.11}$$

where $\widetilde{m}_L(a, b]$ is the number of left focal points in $(a, b]$ of the auxiliary function $\tilde{Y}$.

A corresponding result for the right focal points in $[a, b)$ can be derived by an analogous method to the proof of Proposition 1.2 in [15]. Alternatively, we may use the relationship $m_L(a, b] + \operatorname{rank} X(b) = m_R[a, b) + \operatorname{rank} X(a)$ between the left and right focal points of Y.

Proposition 1.3. *Under the assumptions of Proposition 1.2, for any conjoined bases Y and $\hat{Y}$ of (H) and ($\hat{\mathrm{H}}$) the dual comparative index $\mu^*(Y(t), \hat{Y}(t))$ is piecewise constant and left-continuous on $[a, b]$ and for every $t_0 \in [a, b)$ the multiplicities $m_R(t_0)$, $\widehat{m}_R(t_0)$, and $\widetilde{m}_R(t_0)$ of right focal points of Y, $\hat{Y}$, and $\tilde{Y} := \hat{Z}^{-1} Y$ at t_0 defined through (1.4) satisfy the equality*

$$m_R(t_0) - \widehat{m}_R(t_0) = \widetilde{m}_R(t_0) + \mu^*(Y(t_0), \hat{Y}(t_0)) - \lim_{t \to t_0^+} \mu^*(Y(t), \hat{Y}(t)). \tag{1.12}$$

Moreover, the numbers of right focal points of Y and $\hat{Y}$ in $[a,b)$ are connected by

$$m_R[a,b) - \widehat{m}_R[a,b) = \widetilde{m}_R[a,b) + \mu^*(Y(a), \hat{Y}(a)) - \mu^*(Y(b), \hat{Y}(b)), \tag{1.13}$$

where $\widetilde{m}_R[a,b)$ is the number of right focal points in $[a,b)$ of the auxiliary function $\tilde{Y}$.

Remark 1.4. We note that the symplectic fundamental matrix $\hat{Z}$ of (Ĥ) in Propositions 1.2 and 1.3 has the form $\hat{Z} = (*, \hat{Y})$. Moreover, it is easy to verify (see [11]) that the function $\tilde{Y} := \hat{Z}^{-1} Y$ is a conjoined basis of the transformed linear Hamiltonian system (H̃), whose coefficient matrix

$$\tilde{\mathcal{H}}(t) := \hat{Z}^T(t)\,[\mathcal{H}(t) - \hat{\mathcal{H}}(t)]\,\hat{Z}(t), \quad t \in \mathcal{I}, \tag{1.14}$$

satisfies $\tilde{\mathcal{H}}(t) \geq 0$ on $\mathcal{I}$ under (1.1).

The question regarding the validity of the singular Sturmian comparison theorem for two nonoscillatory linear Hamiltonian systems (H) and (Ĥ) satisfying (1.1) on an unbounded interval $\mathcal{I}$ and no controllability condition is an open problem so far. In the present paper we solve this problem and provide a generalization of Propositions 1.1, 1.2, and 1.3 to this setting (Theorem 5.1). This extension is by no means straightforward. The investigation of this problem revealed the necessity to extend first the transformation theory of linear Hamiltonian systems involving the comparative index, known in [16,17], to unbounded intervals (Theorem 3.2). Along this way we also obtained new results regarding the transformation of the principal and antiprincipal solutions at ∞ (Theorems 3.6, 3.8, and 4.4), which play a fundamental role in our new singular Sturmian comparison theorems.

As we recently observed in [33], when considering an unbounded interval $\mathcal{I} = [a, \infty)$ it is essential to include the multiplicities of focal points at ∞ and to use the minimal principal solutions of (H) and (Ĥ) at ∞ as the reference solutions for counting the focal points. Following [28], the minimal principal solution of (H) at ∞ is defined as the conjoined basis Y_∞ of (H) with $X_\infty(t)$ having constant kernel on $[\alpha, \infty)$ for some $\alpha \in [a, \infty)$ and

$$\lim_{t \to \infty} \left(\int_\alpha^t X_\infty^\dagger(s)\, B(s)\, X_\infty^{\dagger T}(s)\, \mathrm{d}s \right)^\dagger = 0, \quad \operatorname{rank} X_\infty(t) = n - d_\infty, \quad t \in [\alpha, \infty). \tag{1.15}$$

Here † denotes the Moore–Penrose pseudoinverse (see [5–7]), the number d_∞ is the maximal order of abnormality of (H), and the rank of $X_\infty(t)$ is minimal possible on $[\alpha, \infty)$. One can see that condition (1.15) directly generalizes (1.8) to the uncontrollable setting. As applications of the main comparison theorem we derive additional exact formulas and estimates for the numbers of focal points of the principal solutions Y_a, $\hat{Y}_a$ and Y_∞, $\hat{Y}_\infty$ (Theorem 5.5 and Corollaries 5.4 and 5.7). Finally, we note that all the results in this paper are new even for controllable linear Hamiltonian systems, in particular they are also new for the even order Sturm–Liouville differential equations. We are thus convinced that this paper represents an important contribution to the qualitative theory of differential equations. Along the way to the above Sturmian comparison theorems we also discovered the necessity to complete some theoretical results from matrix analysis (Theorem A.2 in the appendix), which were initiated in [22].

The paper is organized as follows. In Section 2 we present the definition and main properties of the comparative index, as well as the needed theory for the known singular Sturmian separation theorems on $\mathcal{I} = [a, \infty)$. In Section 3 we investigate the transformation theory and limit properties of the comparative index with a general symplectic transformation matrix $R(t)$. In Section 4 we apply these results to systems (H) and ($\hat{\text{H}}$) satisfying majorant condition (1.1) and to the special transformation matrix $R(t) := \hat{Z}_\infty(t)$, being the symplectic fundamental matrix of the minorant system ($\hat{\text{H}}$) associated with the minimal principal solution $\hat{Y}_\infty$ of ($\hat{\text{H}}$) at ∞. In Section 5 we present the main results of this paper – singular Sturmian comparison theorems on the unbounded interval $\mathcal{I} = [a, \infty)$, while in Section 6 we present analogous results for the unbounded interval $(-\infty, b]$. Finally, in Appendix A we derive a completion of some known results from matrix analysis related to normalized conjoined bases.

2. Main tools and auxiliary results

The main results of this paper are based on the notion of a comparative index of two conjoined bases of (H) and ($\hat{\text{H}}$). Following [13, Definition 2.1] or [14, Definition 2.1], for two real constant $2n \times n$ matrices Y and $\hat{Y}$ such that

$$Y^T \mathcal{J} Y = 0, \quad \hat{Y}^T \mathcal{J} \hat{Y} = 0, \quad \operatorname{rank} Y = n = \operatorname{rank} \hat{Y}, \quad W := Y^T \mathcal{J} \hat{Y} \tag{2.1}$$

we define their *comparative index* $\mu(Y, \hat{Y})$ and the *dual comparative index* $\mu^*(Y, \hat{Y})$ by

$$\mu(Y, \hat{Y}) := \operatorname{rank} \mathcal{M} + \operatorname{ind} \mathcal{P}, \quad \mu^*(Y, \hat{Y}) := \operatorname{rank} \mathcal{M} + \operatorname{ind}(-\mathcal{P}), \tag{2.2}$$

where $\mathcal{M}$ and $\mathcal{P}$ are the $n \times n$ matrices

$$\mathcal{M} := (I - X^\dagger X)\, W, \quad \mathcal{P} := V\, W^T X^\dagger \hat{X}\, V, \quad V := I - \mathcal{M}^\dagger \mathcal{M}. \tag{2.3}$$

The matrices Y and $\hat{Y}$ are partitioned into the $n \times n$ blocks according to the standard notation $Y = (X^T, U^T)^T$ and $\hat{Y} = (\hat{X}^T, \hat{U}^T)^T$. We note that the matrix V is the orthogonal projector onto $\operatorname{Ker} \mathcal{M}$ and the matrix $\mathcal{P}$ is symmetric, see [14, Theorem 2.1]. The quantity $\operatorname{ind} \mathcal{P}$ denotes the index of $\mathcal{P}$, i.e., the number of its negative eigenvalues. Obviously, the relation $\operatorname{ind}(-\mathcal{P}) = \operatorname{rank} \mathcal{P} - \operatorname{ind} \mathcal{P}$ holds. The needed algebraic properties of the comparative index are summarized as follows, see [14, Section 2].

Proposition 2.1. *Let Y and $\hat{Y}$ be $2n \times n$ matrices satisfying* (2.1) *and let E be given in* (1.6). *Then the comparative index and the dual comparative index defined in* (2.2) *satisfy*

$$\mu(Y, \hat{Y}) + \mu^*(Y, \hat{Y}) = \operatorname{rank} W - \operatorname{rank} X + \operatorname{rank} \hat{X}, \tag{2.4}$$

$$\max\left\{\mu(Y, \hat{Y}),\ \mu^*(Y, \hat{Y})\right\} \le \min\{\operatorname{rank} W,\ \operatorname{rank} \hat{X}\} \le n, \tag{2.5}$$

$$\mu(Y, E) = 0 = \mu^*(Y, E), \quad \mu(E, Y) = \operatorname{rank} X = \mu^*(E, Y), \tag{2.6}$$

$$\mu(YM, \hat{Y}\hat{M}) = \mu(Y, \hat{Y}), \quad \mu^*(YM, \hat{Y}\hat{M}) = \mu^*(Y, \hat{Y}), \quad M, \hat{M} \in \mathbb{R}^{n\times n} \textit{ invertible}. \tag{2.7}$$

Proposition 2.2. *Let V, Z_1, Z_2 be real $2n \times 2n$ symplectic matrices and let E be defined in (1.6). Then the following transformation formulas hold:*

$$\mu(VZ_1E, VE) - \mu(VZ_2E, VE) = \mu(VZ_1E, VZ_2E) - \mu(Z_1E, Z_2E) \tag{2.8}$$

$$= \mu(Z_2E, V^{-1}E) - \mu(Z_1E, V^{-1}E), \tag{2.9}$$

$$\mu^*(VZ_1E, VE) - \mu^*(VZ_2E, VE) = \mu^*(VZ_1E, VZ_2E) - \mu^*(Z_1E, Z_2E) \tag{2.10}$$

$$= \mu^*(Z_2E, V^{-1}E) - \mu^*(Z_1E, V^{-1}E). \tag{2.11}$$

Next we review some background from the oscillation theory of linear Hamiltonian systems. We split the coefficient matrices $\mathcal{H}(t)$, $\hat{\mathcal{H}}(t)$, and $\mathcal{J}$ into $n \times n$ blocks as

$$\mathcal{H}(t) = \begin{pmatrix} -C(t) & A^T(t) \\ A(t) & B(t) \end{pmatrix}, \quad \hat{\mathcal{H}}(t) = \begin{pmatrix} -\hat{C}(t) & \hat{A}^T(t) \\ \hat{A}(t) & \hat{B}(t) \end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{2.12}$$

where $A(t)$, $B(t)$, $C(t)$ and $\hat{A}(t)$, $\hat{B}(t)$, $\hat{C}(t)$ are piecewise continuous on $\mathcal{I}$ and $B(t)$, $C(t)$, $\hat{B}(t)$, $\hat{C}(t)$ are symmetric for $t \in \mathcal{I}$. From (1.1) and (1.2) it follows that the majorant system (H) satisfies the corresponding Legendre condition

$$B(t) \geq 0 \quad \text{for all } t \in \mathcal{I} \tag{2.13}$$

as well. By $A \geq 0$ we mean that the symmetric matrix A is positive semidefinite and for symmetric matrices A and B the notation $A \geq B$ means that $A - B \geq 0$. Similarly to (2.12) we will split the matrix solutions of (H) and ($\hat{\text{H}}$) into their $n \times n$-matrix components as

$$Y(t) = \begin{pmatrix} X(t) \\ U(t) \end{pmatrix}, \quad Y_{t_0}(t) = \begin{pmatrix} X_{t_0}(t) \\ U_{t_0}(t) \end{pmatrix}, \quad \hat{Y}(t) = \begin{pmatrix} \hat{X}(t) \\ \hat{U}(t) \end{pmatrix}, \quad \hat{Y}_{t_0}(t) = \begin{pmatrix} \hat{X}_{t_0}(t) \\ \hat{U}_{t_0}(t) \end{pmatrix}. \tag{2.14}$$

We will be particularly interested in *conjoined bases* of (H) and ($\hat{\text{H}}$), i.e., the solutions Y satisfying $Y^T(t)\,\mathcal{J}\,Y(t) = 0$ and $\operatorname{rank} Y(t) = n$ at some (and hence at all) points $t \in \mathcal{I}$.

Next we recall several important results regarding the unbounded interval $\mathcal{I} = [a, \infty)$ and the corresponding Sturmian separation theorems from [33]. System (H) is defined to be *nonoscillatory at* ∞ if for some conjoined basis Y of (H) (or for every conjoined basis Y of (H) by [35, Theorem 2.2]) there are no left focal points of Y in (α, ∞) for some $\alpha \in [a, \infty)$. In this case the number α can be chosen so that the kernel of $X(t)$ is constant on $[\alpha, \infty)$. With a slight abuse in the terminology we will say in this case that the conjoined basis Y itself has a constant kernel on $[\alpha, \infty)$. Then by [23, Lemma 2] we also have

$$X(t)\,X^\dagger(t)\,B(t) = B(t) = B(t)\,X(t)\,X^\dagger(t), \quad t \in [\alpha, \infty). \tag{2.15}$$

Note that we may equivalently define the nonoscillation of Y at ∞ in terms of the nonexistence of the right focal points in $[\alpha, \infty)$, since by [31, Theorem 5.1] we have the equality

$$m_L(\alpha, \beta] + \operatorname{rank} X(\beta) = m_R[\alpha, \beta) + \operatorname{rank} X(\alpha), \quad \alpha, \beta \in [a, \infty),\ \alpha < \beta. \tag{2.16}$$

In particular, if $\operatorname{rank} X(t)$ is constant on $[\alpha, \infty)$, then (2.16) yields $m_L(\alpha, \infty) = m_R[\alpha, \infty)$.

The maximal order of abnormality of (H) at ∞ is defined as the number

$$d_\infty := \lim_{t\to\infty} d[t,\infty) = \max_{t\in[a,\infty)} d[t,\infty), \quad 0 \le d_\infty \le n, \tag{2.17}$$

where $d[t,\infty)$ is the order of abnormality of system (H) on $[t,\infty)$, i.e., $d[t,\infty)$ is the dimension of the space of vector solutions $(x \equiv 0, u)$ of (H) on the interval $[t,\infty)$. For a completely controllable system (H) we have $d[t,\infty) = d_\infty = 0$ for all $t \in [a,\infty)$. The number d_∞ is one of the important parameters of system (H). For example, in [28,30,32] we showed that every conjoined basis Y of (H) with constant kernel on $[\alpha,\infty)$ with $d[\alpha,\infty) = d_\infty$ satisfies

$$n - d_\infty \le \operatorname{rank} X(t) \le n, \quad t \in [\alpha,\infty), \tag{2.18}$$

$$T_{\alpha,\infty} := \lim_{t\to\infty} \Big(\int_\alpha^t X^\dagger(s)\, B(s)\, X^{\dagger T}(s)\, \mathrm{d}s \Big)^\dagger, \quad 0 \le \operatorname{rank} T_{\alpha,\infty} \le n - d_\infty, \tag{2.19}$$

where the limit exists due to (2.13) and the matrix $T_{\alpha,\infty}$ is symmetric and positive semidefinite. Then, according to [29, Definition 7.1], a conjoined basis Y of (H) with constant kernel on $[\alpha,\infty)$ with $d[\alpha,\infty) = d_\infty$ is a *principal solution at* ∞ if the corresponding matrix in (2.19) satisfies $T_{\alpha,\infty} = 0$, i.e., the rank of $T_{\alpha,\infty}$ is minimal. And according to [30, Definition 5.1], a conjoined basis Y of (H) with constant kernel on $[\alpha,\infty)$ with $d[\alpha,\infty) = d_\infty$ is an *antiprincipal solution at* ∞ if the corresponding matrix in (2.19) satisfies $\operatorname{rank} T_{\alpha,\infty} = n - d_\infty$, i.e., the rank of $T_{\alpha,\infty}$ is maximal. In addition, if $\operatorname{rank} X(t) = n - d_\infty$ holds, then the (anti)principal solution of (H) at ∞ is called *minimal* (compare with (1.15)), while if eventually $\operatorname{rank} X(t) = n$ holds, then the (anti)principal solution of (H) at ∞ is called *maximal*. This terminology complies with the estimates in (2.18) for the rank of $X(t)$, see [29,30,32] for more details. We note that the minimal principal solution of (H) at ∞ is uniquely determined up to a nonsingular right multiple, see [28, Theorem 7.6] and compare with [29, Theorem 7.6] and [30, Theorem 5.8].

For nonoscillatory systems (H) and ($\hat{\mathrm{H}}$) at ∞ we will denote their (unique) minimal principal solutions at ∞ by Y_∞ and $\hat{Y}_\infty$, respectively. This notation complies with the notation for the principal solutions at a finite point t_0 in (2.14), since these principal solutions play an analogous role in the Sturmian theory, see [32, Theorem 5.8].

In [33, Definition 3.1] we introduced the *multiplicity of a (left) focal point at* ∞ of a conjoined basis Y of (H) with constant kernel on $[\alpha,\infty)$ with $d[\alpha,\infty) = d_\infty$ by

$$m_L(\infty) := n - d_\infty - \operatorname{rank} T_{\alpha,\infty}, \tag{2.20}$$

where the matrix $T_{\alpha,\infty}$ is defined in (2.19). Moreover, by [33, Theorem 3.3] the number $m_L(\infty)$ is related to the minimal principal solution Y_∞ of (H) at ∞ by the formula

$$m_L(\infty) = \lim_{t\to\infty} \operatorname{rank} X(t) - \operatorname{rank} W(Y_\infty, Y). \tag{2.21}$$

By [32, Theorem 6.1], this property of $m_L(\infty)$ corresponds to the multiplicity at a finite point t_0 presented in (1.3). Expression (2.20) shows that for principal solutions of (H) at ∞ (in particular, for Y_∞) we have $m_L(\infty) = n - d_\infty$ (i.e., the multiplicity at ∞ is maximal), while for antiprincipal solutions of (H) at ∞ we have $m_L(\infty) = 0$ (i.e., the multiplicity at ∞ is minimal). From (2.21) we then obtain the following result.

Proposition 2.3. *Assume that* (2.13) *holds with* $\mathcal{I} = [a, \infty)$ *and system* (H) *is nonoscillatory at* ∞. *Then a conjoined basis* Y *of* (H) *is an antiprincipal solution at* ∞ *if and only if*

$$\lim_{t\to\infty} \operatorname{rank} X(t) = \operatorname{rank} W(Y_\infty, Y).$$

In particular, the conjoined basis Y *is a maximal antiprincipal solution at* ∞ *if and only if the Wronskian* $W(Y_\infty, Y)$ *is invertible.*

In [33, Section 5] we derived the singular Sturmian *separation* theorems on unbounded intervals $\mathcal{I}$ for conjoined bases of one linear Hamiltonian system (H). These results are based on the representation of conjoined bases of (H) in terms of a suitable fundamental matrix. Specifically, we consider the symplectic fundamental matrix Z_∞ of (H) associated with the minimal principal solution Y_∞ of (H) at ∞ through the equality $Y_\infty(t) = Z_\infty(t)\, E$ on $\mathcal{I}$, that is,

$$Z_\infty(t) = \begin{pmatrix} \bar{Y}_\infty(t) & Y_\infty(t) \end{pmatrix}, \quad t \in \mathcal{I}, \quad W(\bar{Y}_\infty, Y_\infty) = I. \tag{2.22}$$

Here $\bar{Y}_\infty$ is a conjoined basis of (H) completing Y_∞ to a pair of normalized conjoined bases, see e.g. [22, Proposition 4.1.1]. Then every conjoined basis Y of (H) can be uniquely represented by a constant $2n \times n$ matrix C_∞ satisfying

$$Y(t) = Z_\infty(t)\, C_\infty, \quad t \in \mathcal{I}, \quad C_\infty = \begin{pmatrix} -W(Y_\infty, Y) \\ W(\bar{Y}_\infty, Y) \end{pmatrix}. \tag{2.23}$$

Observe that the results below include the multiplicities of focal points at ∞, according to the notation introduced in (1.5).

Proposition 2.4 *(Singular Sturmian separation theorem). Assume that* (2.13) *holds on the interval* $\mathcal{I} = [a, \infty)$ *and system* (H) *is nonoscillatory at* ∞. *Then for any conjoined bases* Y *and* $\hat{Y}$ *of* (H) *we have the equalities*

$$m_L(a, \infty] - \widehat{m}_L(a, \infty] = \mu(C_\infty, \hat{C}_\infty) - \mu(Y(a), \hat{Y}(a)), \tag{2.24}$$

$$= \mu(\hat{Y}(a), Y_\infty(a)) - \mu(Y(a), Y_\infty(a)), \tag{2.25}$$

$$m_R[a, \infty) - \widehat{m}_R[a, \infty) = \mu^*(Y(a), \hat{Y}(a)) - \mu^*(C_\infty, \hat{C}_\infty), \tag{2.26}$$

$$= \mu^*(Y(a), Y_\infty(a)) - \mu^*(\hat{Y}(a), Y_\infty(a)), \tag{2.27}$$

where C_∞ *and* $\hat{C}_\infty$ *are the constant matrices in* (2.23) *corresponding to* Y *and* $\hat{Y}$.

Proof. Formulas (2.24) and (2.26) were proven in [33, Theorem 5.1]. Next we use (2.24) for the conjoined bases Y and Y_∞ (i.e., with the representation matrices C_∞ and E), and then we use (2.24) again for the conjoined bases $\hat{Y}$ and Y_∞ (i.e., with the representation matrices $\hat{C}_\infty$ and E). Subtracting the resulting equalities yields formula (2.25). Similarly, applying (2.26) once to Y and Y_∞ and once to $\hat{Y}$ and Y_∞ and subtracting the outcome leads to formula (2.27). □

Remark 2.5. We note that in [33] we used the alternative symplectic fundamental matrix $\Phi_\infty = (Y_\infty, *)$ and the representation $Y(t) = \Phi_\infty(t)\, D_\infty$ on $\mathcal{I}$, which yields that $C_\infty = -\mathcal{J} D_\infty$. But since by property (2.7) with $M = -I = \hat{M}$ we have $\mu(\mathcal{J}D_\infty, \mathcal{J}\hat{D}_\infty) = \mu(C_\infty, \hat{C}_\infty)$ and $\mu^*(\mathcal{J}D_\infty, \mathcal{J}\hat{D}_\infty) = \mu^*(C_\infty, \hat{C}_\infty)$, all the formulas in [33] involving the comparative index (or the dual comparative index) with $\mathcal{J}D_\infty$ can be replaced by the same formulas involving the matrices C_∞.

For the special choice of $\hat{Y} := Y_a$ (being the principal solution at a) or $\hat{Y} := Y_\infty$ (being the minimal principal solution at ∞) the results in Proposition 2.4 yield the following, see [33, Corollary 5.3 and Theorems 5.5 and 5.7].

Proposition 2.6 *(Singular Sturmian separation theorem). Assume that (2.13) holds on the interval $\mathcal{I} = [a, \infty)$ and system (H) is nonoscillatory at ∞. Then for any conjoined basis Y of (H) we have the equalities*

$$m_{La}(a, \infty] + \mu(C_\infty, C_\infty^a) = m_L(a, \infty] = m_{L\infty}(a, \infty] - \mu(Y(a), Y_\infty(a)), \quad (2.28)$$

$$m_{R\infty}[a, \infty) + \mu^*(Y(a), Y_\infty(a)) = m_R[a, \infty) = m_{Ra}[a, \infty) - \mu^*(C_\infty, C_\infty^a), \quad (2.29)$$

where C_∞ and C_∞^a are the constant matrices in (2.23) corresponding to Y and Y_a. Moreover,

$$m_{La}(a, \infty] \le m_L(a, \infty] \le m_{L\infty}(a, \infty], \quad (2.30)$$

$$m_{R\infty}[a, \infty) \le m_R[a, \infty) \le m_{Ra}[a, \infty), \quad (2.31)$$

and the above lower and upper bounds are related by the equalities

$$m_{L\infty}(a, \infty] = m_{La}(a, \infty] + \operatorname{rank} X_\infty(a), \qquad m_{Ra}[a, \infty) = m_{R\infty}[a, \infty) + \operatorname{rank} X_\infty(a), \quad (2.32)$$

$$m_{La}(a, \infty] = m_{R\infty}[a, \infty), \qquad m_{L\infty}(a, \infty] = m_{Ra}[a, \infty). \quad (2.33)$$

Finally, for completeness we present a formula relating the numbers of left and right focal points of one conjoined basis of (H) in an unbounded interval, see [33, Theorems 5.4]. This formula also involves the minimal principal solution Y_∞ at ∞, as well as the principal solution Y_a at the left endpoint a, since $X(a) = -W(Y_a, Y)$.

Proposition 2.7. *Assume that (2.13) holds with $\mathcal{I} = [a, \infty)$ and system (H) is nonoscillatory at ∞. Then for any conjoined basis Y of (H) we have the equality*

$$m_L(a, \infty] + \operatorname{rank} W(Y_\infty, Y) = m_R[a, \infty) + \operatorname{rank} X(a). \quad (2.34)$$

Formula (2.34) will be used in the proof of the singular comparison theorem for the right focal points, knowing the result for the left focal points. The results in Propositions 2.4–2.7 highlight the symmetric role of the (minimal) principal solutions Y_a and Y_∞ in the Sturmian separation theorems for one system (H) on the interval $\mathcal{I} = [a, \infty)$. Similar symmetry holds also for other types of unbounded intervals $\mathcal{I} = (-\infty, b]$ or $\mathcal{I} = (-\infty, \infty)$ and the (minimal) principal solutions at their endpoints, see [33, Remark 8.1].

3. Transformation and limit results for comparative index

In this section we consider a nonoscillatory linear Hamiltonian system (H) on $\mathcal{I} = [a, \infty)$ and a given piecewise continuously differentiable $2n \times 2n$ symplectic transformation matrix $R(t)$ on the interval $\mathcal{I}$. Analogously to Remark 1.4, the transformation

$$\tilde{y} := R^{-1}(t)\, y \tag{3.1}$$

transforms system (H) into linear Hamiltonian system ($\tilde{\text{H}}$) with the coefficient matrix

$$\tilde{\mathcal{H}}(t) := R^T(t)\, [\mathcal{H}(t) - \mathcal{H}_R(t)]\, R(t), \quad \mathcal{H}_R(t) := \mathcal{J} R'(t)\, \mathcal{J} R^T(t)\, \mathcal{J}, \quad t \in \mathcal{I}. \tag{3.2}$$

We will assume in this section that the transformed system ($\tilde{\text{H}}$) is nonoscillatory at ∞ and satisfies the Legendre condition

$$\tilde{B}(t) \geq 0 \quad \text{for all } t \in \mathcal{I}, \quad \tilde{B}(t) := E^T\, \tilde{\mathcal{H}}(t)\, E, \tag{3.3}$$

with the matrix E given in (1.6). Our aim is to establish for a given conjoined basis Y of (H) limit results at ∞ for the comparative indices

$$\mu(Y(t), R(t)E) \quad \text{and} \quad \mu^*(Y(t), R(t)E), \tag{3.4}$$

and to find conditions under which the minimal principal solution of (H) at ∞ transforms to the minimal principal solution of ($\tilde{\text{H}}$) at ∞.

Since the matrix $R(t)$ is piecewise continuously differentiable and symplectic on $\mathcal{I}$, the matrix function $R(t)E$ appearing in (3.4) is a conjoined basis of the linear Hamiltonian system

$$y' = \mathcal{J}\mathcal{H}_R(t)\, y, \quad t \in \mathcal{I}, \tag{H_R}$$

with the symmetric coefficient matrix $\mathcal{H}_R(t)$ defined in (3.2). When $R(t)$ is a fundamental matrix of the original system (H), then of course $\mathcal{H}_R(t) \equiv \mathcal{H}(t)$ and $R(t)E$ is a conjoined basis of (H). In this case the limit results at ∞ for (3.4) are known in [33, Theorem 6.1].

Proposition 3.1. *Assume that (2.13) holds with $\mathcal{I} = [a, \infty)$ and system (H) is nonoscillatory at ∞. Let Y be a conjoined basis of (H) and let $R(t)$ be a symplectic fundamental matrix of (H). Then for $Y^*(t) := R(t)E$ we have*

$$\lim_{t\to\infty} \mu(Y(t), Y^*(t)) = \mu(C_\infty, C_\infty^*) - m_L(\infty) + m_L^*(\infty), \tag{3.5}$$

$$\lim_{t\to\infty} \mu^*(Y(t), Y^*(t)) = \mu^*(C_\infty, C_\infty^*), \tag{3.6}$$

where C_∞ and C_∞^* are the constant matrices in (2.23) corresponding to Y and Y^*, and where $m_L^*(\infty)$ is the multiplicity of a focal point at ∞ of the conjoined basis Y^*.

In the main result of this section (Theorem 3.2 below) we generalize Proposition 3.1 to an arbitrary piecewise continuously differentiable symplectic matrix $R(t)$. For convenience and in accordance with [17, Eq. (1.3)] we split the matrix $R(t)$ into the $n \times n$ blocks as

$$R(t) = \begin{pmatrix} L(t) & M(t) \\ K(t) & N(t) \end{pmatrix}, \quad R(t)E = \begin{pmatrix} M(t) \\ N(t) \end{pmatrix}, \quad t \in \mathcal{I}. \tag{3.7}$$

We recall the notation Y_∞ and $\tilde{Y}_\infty$ for the minimal principal solutions of (H) and ($\tilde{\text{H}}$) at ∞.

Theorem 3.2. *Assume that* (2.13) *holds with* $\mathcal{I} = [a, \infty)$*, system* (H) *is nonoscillatory at* ∞*, and* Y *is a conjoined basis of* (H)*. Let* $R(t)$ *be a piecewise continuously differentiable symplectic matrix on* $\mathcal{I}$ *with the partition in* (3.7) *and assume that system* ($\tilde{\text{H}}$) *with* (3.2) *satisfies* (3.3) *and it is nonoscillatory at* ∞*. Then the limits*

$$\lim_{t\to\infty} \mu(Y(t), R(t)E) \quad \text{and} \quad \lim_{t\to\infty} \mu^*(Y(t), R(t)E) \tag{3.8}$$

both exist if and only if

$$\operatorname{rank} M(t) \text{ is eventually constant,} \tag{3.9}$$

i.e., the limit of $\operatorname{rank} M(t)$ *exists for* $t \to \infty$*. In this case the limits*

$$\lim_{t\to\infty} \mu^*\big(R^{-1}(t)\, Y_\infty(t), R^{-1}(t)E\big) \quad \text{and} \quad \lim_{t\to\infty} \mu^*\big(R(t)\, \tilde{Y}_\infty(t), R(t)E\big) \tag{3.10}$$

also exist and we have the equalities

$$\begin{aligned} \lim_{t\to\infty} \mu(Y(t), R(t)E) &= \mu(C_\infty, C_{\infty,R}) - m_L(\infty) + \widetilde{m}_L(\infty) \\ &\qquad + \lim_{t\to\infty} \mu^*\big(R^{-1}(t)\, Y_\infty(t), R^{-1}(t)E\big), \end{aligned} \tag{3.11}$$

$$\lim_{t\to\infty} \mu^*(Y(t), R(t)E) = \mu^*(C_\infty, C_{\infty,R}) + \lim_{t\to\infty} \mu^*\big(R(t)\, \tilde{Y}_\infty(t), R(t)E\big). \tag{3.12}$$

Here C_∞ *and* $C_{\infty,R}$ *are the constant matrices in* (2.23) *corresponding to* Y *and* $R\tilde{Y}_\infty$*, and* $\widetilde{m}_L(\infty)$ *is the multiplicity of a focal point at* ∞ *of the conjoined basis* $\tilde{Y} := R^{-1}Y$ *of the transformed system* ($\tilde{\text{H}}$)*.*

The remaining part of this section (except of the last result presented in Corollary 3.11) will be devoted to developing the necessary tools for the proof of Theorem 3.2. First we present several comments and needed results, which turn to be important on their own independently of their future applications in Sections 4 and 5.

Remark 3.3. (i) Condition (3.9) in Theorem 3.2 is independent of Y. Therefore, when the limits in (3.8) exist for one conjoined basis Y of (H), then these limits exist for every conjoined basis Y of (H). Moreover, condition (3.9) guarantees through [16, Theorem 2.2] that the oscillation properties of systems (H) and ($\tilde{\text{H}}$) are preserved, that is, under (2.13) and (3.3) systems (H) and ($\tilde{\text{H}}$) oscillate or do not oscillate at ∞ simultaneously.

(ii) In [17, Theorem 2.5 and Remark 2.6(ii)], the existence of the limits in (3.8) at a finite point $t_0 \in \mathcal{I}$ is proven under the sufficient condition that the rank of $M(t)$ is constant in a left and right deleted neighborhood of t_0. The statement in Theorem 3.2 can be regarded as an analogue of those results for $t_0 = \infty$, namely formula (3.11) corresponds to [17, Eq. (2.11)] and formula (3.12) corresponds to the second part of [17, Eq. (2.17)]. At the same time, the proof of Theorem 3.2 will show that the stated condition on the constant rank of $M(t)$ in [17, Theorem 2.5] is not only sufficient, but also necessary.

(iii) If in addition the system (H_R) satisfies the Legendre condition $B_R(t) := E^T \mathcal{H}_R(t)\, E \geq 0$ for all $t \in \mathcal{I} = [a, \infty)$, then condition (3.9) is equivalent with $\operatorname{Ker} M(t)$ being eventually constant, and hence (3.9) is equivalent with the nonoscillation of system (H_R) at ∞.

For a conjoined basis Y of (H) and a point $t \in \mathcal{I}$ we define the quantities

$$q(Y,t) := \mu(Y(t), R(t)E) - \mu(C_\infty, C_{\infty,R}) + m_L(\infty) - \widetilde{m}_L(\infty), \tag{3.13}$$

$$q^*(Y,t) := \mu^*(Y(t), R(t)E) - \mu^*(C_\infty, C_{\infty,R}), \tag{3.14}$$

where $m_L(\infty)$ and $\widetilde{m}_L(\infty)$ are the multiplicities of the focal point at ∞ of Y and $\tilde{Y} := R^{-1}Y$, and where C_∞ and $C_{\infty,R}$ are the constant matrices in (2.23) corresponding to Y and $R\,\tilde{Y}_\infty$. By using property (2.4) of the comparative index and formula (2.21) for the multiplicity of a focal point at ∞ we then obtain

$$\lim_{t\to\infty} \big\{\operatorname{rank} M(t) - q(Y,t) - q^*(Y,t)\big\} = \operatorname{rank} W(Y_\infty, R\,\tilde{Y}_\infty). \tag{3.15}$$

This formula represents an analogue of [17, Eq. (2.16)] for the case of $t_0 = \infty$. The quantities $q(Y,t)$ and $q^*(Y,t)$ allow to interpret Theorem 3.2 in a simpler form and to shorten its proof.

Remark 3.4. With the aid of the quantities $q(Y,t)$ and $q^*(Y,t)$ defined in (3.13) and (3.14) we may reformulate the result in Theorem 3.2 as follows. The limits

$$\lim_{t\to\infty} q(Y,t) \quad \text{and} \quad \lim_{t\to\infty} q^*(Y,t) \tag{3.16}$$

both exist if and only if the rank of $M(t)$ is eventually constant, and in this case

$$\lim_{t\to\infty} q(Y,t) = \lim_{t\to\infty} \mu^*\big(R^{-1}(t)\,Y_\infty(t),\, R^{-1}(t)E\big), \tag{3.17}$$

$$\lim_{t\to\infty} q^*(Y,t) = \lim_{t\to\infty} \mu^*\big(R(t)\,\tilde{Y}_\infty(t),\, R(t)E\big). \tag{3.18}$$

Formulas (3.17) and (3.18) confirm that the values of the limits in (3.16) do not depend on the choice of the conjoined basis Y of (H).

Remark 3.5. When $R(t)$ is a symplectic fundamental matrix of system (H), then the matrix $\mathcal{H}_R(t) = \mathcal{H}(t)$ in (3.2), and hence $\tilde{\mathcal{H}}(t) \equiv 0$. In this case all solutions of system ($\tilde{\mathrm{H}}$) are constant on $\mathcal{I}$, in particular $\tilde{Y}_\infty(t) \equiv E$ and $\widetilde{m}_L(\infty) = 0$ (compare with [28, Example 8.2] and [33, Remark 3.2(ii)]). Then we have

$$\lim_{t\to\infty} \mu^*\big(R^{-1}(t)\,Y_\infty(t),\, R^{-1}(t)E\big) = m_L^*(\infty), \quad \lim_{t\to\infty} \mu^*\big(R(t)\,\tilde{Y}_\infty(t),\, R(t)E\big) = 0, \tag{3.19}$$

where $m_L^*(\infty)$ is the multiplicity of a focal point at ∞ of the conjoined basis $Y^* := R\,E$ of (H). This can be seen as follows. The second limit in (3.19) holds trivially. For the first limit we apply property (2.11) (with $V := R^{-1}(t)$, $Z_1 := Z_\infty(t)$, and $Z_2 := I$) to get

$$\mu^*\big(R^{-1}(t)\,Y_\infty(t),\, R^{-1}(t)E\big) = \operatorname{rank} M(t) - \mu^*(Y_\infty(t),\, R(t)E), \quad t \in \mathcal{I}. \tag{3.20}$$

Then by taking the limit for $t \to \infty$ and using equality (3.6) in Proposition 3.1 (with $Y := Y_\infty$ and $C_\infty := E$), we obtain

$$\begin{aligned}\lim_{t\to\infty} \mu^*\big(R^{-1}(t)\,Y_\infty(t),\, R^{-1}(t)E\big) &= \lim_{t\to\infty} \operatorname{rank} M(t) - \mu^*(E, C_\infty^*)\\ &\overset{(2.6),(2.23)}{=} \lim_{t\to\infty} \operatorname{rank} M(t) - \operatorname{rank} W(Y_\infty, Y^*) \overset{(2.21)}{=} m_L^*(\infty).\end{aligned}$$

This shows that when $R(t)$ is a symplectic fundamental matrix of system (H), then Theorem 3.2 reduces exactly to Proposition 3.1. Note also that in this case (3.17) and (3.18) have the form

$$\lim_{t\to\infty} q(Y,t) = m_L^*(\infty), \qquad \lim_{t\to\infty} q^*(Y,t) = 0. \tag{3.21}$$

In the proof of Theorem 3.2 we will utilize a similar representation of conjoined bases of the transformed system ($\tilde{\text{H}}$) as in (2.22) and (2.23). That is, we consider the symplectic fundamental matrix $\tilde{Z}_\infty$ of ($\tilde{\text{H}}$) associated with the minimal principal solution $\tilde{Y}_\infty$ of ($\tilde{\text{H}}$) at ∞ through the equality $\tilde{Y}_\infty(t) = \tilde{Z}_\infty(t)\,E$ on $\mathcal{I}$. Then every conjoined basis $\tilde{Y}$ of ($\tilde{\text{H}}$) can be uniquely represented by a constant $2n \times n$ matrix $\tilde{C}_\infty$ satisfying

$$\tilde{Y}(t) = \tilde{Z}_\infty(t)\,\tilde{C}_\infty, \quad t \in \mathcal{I}, \quad (I,\ 0)\,\tilde{C}_\infty = -W(\tilde{Y}_\infty, \tilde{Y}). \tag{3.22}$$

In the next statement we consider the transformation of maximal antiprincipal solutions of (H) at ∞ under (3.1). In particular, we describe when a maximal antiprincipal solution of (H) at ∞ is transformed into a maximal antiprincipal solution of ($\tilde{\text{H}}$) at ∞.

Theorem 3.6. *Assume that (2.13) holds with $\mathcal{I} = [a, \infty)$, system (H) is nonoscillatory at ∞, and Y is a conjoined basis of (H). Let $R(t)$ be a piecewise continuously differentiable symplectic matrix on $\mathcal{I}$ and assume that system ($\tilde{\text{H}}$) with (3.2) satisfies (3.3) and it is nonoscillatory at ∞. Moreover, let C_∞ and $C_{\infty,R}$ be the constant matrices in (2.23) corresponding to Y and $R\,\tilde{Y}_\infty$, and let $F_{\infty,R}$ and $D_{\infty,R}$ be the matrices in (A.2) and (A.3) in Appendix A corresponding to the constant $2n \times n$ matrix $C_{\infty,R}$. Then the following statements are equivalent.*

(i) *The conjoined bases Y and $\tilde{Y} := R^{-1}Y$ are maximal antiprincipal solutions at ∞ of systems (H) and ($\tilde{\text{H}}$), respectively.*

(ii) *The matrix C_∞ has the form*

$$C_\infty = (-\mathcal{J} C_{\infty,R}\, F_{\infty,R}^{-1} + C_{\infty,R}\, D)\, K, \tag{3.23}$$

where D and K are constant $n \times n$ matrices such that K is invertible, D is symmetric,

$$\operatorname{Im}\big\{W(Y_\infty, R\,\tilde{Y}_\infty)\,(D - D_{\infty,R})\,[\,W(Y_\infty, R\,\tilde{Y}_\infty)]^T\big\} = \operatorname{Im} W(Y_\infty, R\,\tilde{Y}_\infty). \tag{3.24}$$

In this case $K = W(\tilde{Y}_\infty, \tilde{Y})$, the matrix $W(Y(t), R(t)E)$ is invertible for large $t \in \mathcal{I}$, and

$$\left.\begin{aligned} &\operatorname{ind}\left\{W(Y_\infty, R\tilde{Y}_\infty)(D - D_{\infty,R})\,[W(Y_\infty, R\tilde{Y}_\infty)]^T\right\} \\ &\qquad\qquad = \operatorname{ind}\left\{W(\tilde{Y}_\infty, \tilde{Y})\,[W(Y_\infty, Y)]^{-1}\,W(Y_\infty, R\tilde{Y}_\infty)\right\}. \end{aligned}\right\} \tag{3.25}$$

Proof. The proof is based on the characterization of a maximal antiprincipal solution at ∞ in Proposition 2.3. More precisely, the conjoined bases Y and $\tilde{Y} := R^{-1}Y$ are maximal antiprincipal solutions at ∞ of systems (H) and ($\tilde{\text{H}}$), respectively, if and only if the Wronskians $W(Y_\infty, Y)$ and $W(\tilde{Y}_\infty, \tilde{Y})$ are invertible. Since the matrices $Z_\infty^{-1}(t)$ and $R(t)$ are symplectic on $\mathcal{I}$, we obtain (suppressing the argument $t \in \mathcal{I}$)

$$\begin{aligned} W(C_{\infty,R}, C_\infty) &= C_{\infty,R}^T \mathcal{J} C_\infty \overset{(2.23)}{=} \tilde{Y}_\infty^T R^T Z_\infty^{T\,-1} \mathcal{J} Z_\infty^{-1} Y = \tilde{Y}_\infty^T R^T \mathcal{J} R R^{-1} Y \\ &= \tilde{Y}_\infty^T \mathcal{J} R^{-1} Y = W(\tilde{Y}_\infty, R^{-1}Y) = W(\tilde{Y}_\infty, \tilde{Y}). \end{aligned} \tag{3.26}$$

Moreover, since by (2.23) we have $(I,\ 0)\, C_\infty = -W(Y_\infty, Y)$ and $(I,\ 0)\, C_{\infty,R} = -W(Y_\infty, R\tilde{Y}_\infty)$, the upper $n \times n$ block of C_∞ is invertible. Therefore, $C_{\infty,R}$ and $C_\infty\,[W(\tilde{Y}_\infty, \tilde{Y})]^{-1}$ are constant matrices satisfying the properties required in Theorem A.2 in the appendix (with $Y := C_{\infty,R}$ and $\hat{Y} := C_\infty\,[W(\tilde{Y}_\infty, \tilde{Y})]^{-1}$). Hence, Theorem A.2 yields that statement (i) is equivalent with the fact that C_∞ has the form in (3.23) with $K = W(\tilde{Y}_\infty, \tilde{Y})$ and D satisfying (3.24). Finally, in this case (3.25) holds by (A.5) in Theorem A.2, and the matrix

$$W(Y(t), R(t)E) = W(R^{-1}(t)\,Y(t), E) = W(\tilde{Y}(t), E) = \tilde{X}^T(t) \tag{3.27}$$

is invertible for large $t \in \mathcal{I}$, since $\tilde{Y}$ is a maximal conjoined basis of ($\tilde{\text{H}}$). □

Remark 3.7. Based on Remark A.3(ii) in the appendix we may conclude that for every transformation matrix $R(t)$ in Theorem 3.6 there always exists a maximal antiprincipal solution Y of (H) at ∞, which is transformed to the maximal antiprincipal solution $\tilde{Y} := R^{-1}Y$ of ($\tilde{\text{H}}$) at ∞. For the associated representation matrices C_∞ and $C_{\infty,R}$ of Y and $R\tilde{Y}_\infty$ in Theorem 3.6 we then have, by the definition in (2.2),

$$\mu(C_\infty, C_{\infty,R}) = \operatorname{ind}\left\{-W(\tilde{Y}_\infty, \tilde{Y})\,[W(Y_\infty, Y)]^{-1}\,W(Y_\infty, R\tilde{Y}_\infty)\right\}, \tag{3.28}$$

$$\mu^*(C_\infty, C_{\infty,R}) = \operatorname{ind}\left\{W(\tilde{Y}_\infty, \tilde{Y})\,[W(Y_\infty, Y)]^{-1}\,W(Y_\infty, R\tilde{Y}_\infty)\right\}. \tag{3.29}$$

In these calculations we used that $W(C_\infty, C_{\infty,R}) = -[W(\tilde{Y}_\infty, \tilde{Y})]^T$ obtained from (3.26).

The statement in Theorem 3.6 yields the following result regarding the transformation of maximal antiprincipal solutions of (H) and ($\tilde{\text{H}}$) at ∞, as well as the transformation of minimal principal solutions of (H) and ($\tilde{\text{H}}$) at ∞.

Theorem 3.8. *Assume that (2.13) holds with $\mathcal{I} = [a, \infty)$, system (H) is nonoscillatory at ∞, and Y is a conjoined basis of (H). Let $R(t)$ be a piecewise continuously differentiable symplectic matrix on $\mathcal{I}$ and assume that system ($\tilde{\text{H}}$) with (3.2) satisfies (3.3) and it is nonoscillatory at ∞. Then the following statements are equivalent.*

(i) *Every maximal antiprincipal solution of* (H) *at* ∞ *is transformed into a maximal antiprincipal solution of* ($\tilde{\mathrm{H}}$) *at* ∞.
(ii) *Every maximal antiprincipal solution of* ($\tilde{\mathrm{H}}$) *at* ∞ *is a transformation of some maximal antiprincipal solution of* (H) *at* ∞.
(iii) *The minimal principal solution of* (H) *at* ∞ *is transformed into the minimal principal solution of* ($\tilde{\mathrm{H}}$) *at* ∞.
(iv) *The constant Wronskian matrices* $W(R^{-1}Y_\infty, \tilde{Y}_\infty)$ *and* $W(Y_\infty, R\tilde{Y}_\infty)$ *satisfy*

$$W(R^{-1}Y_\infty, \tilde{Y}_\infty) = 0 = W(Y_\infty, R\tilde{Y}_\infty).$$

Proof. Clearly, conditions (iii) and (iv) are equivalent. Next we show that (i) is equivalent with condition $W(Y_\infty, R\tilde{Y}_\infty) = 0$ in (iv). Assume that (i) holds. Then by Theorem 3.6 condition (3.24) is satisfied for every symmetric matrix D. In particular, for $D := D_{\infty,R}$ in (3.24) we obtain that $W(Y_\infty, R\tilde{Y}_\infty) = 0$. Conversely, if (iv) holds, then condition (3.24) is trivially satisfied for every symmetric matrix D, so that condition (i) follows from Theorem 3.6. Finally, in a similar way we prove that (ii) is equivalent with condition $W(R^{-1}Y_\infty, \tilde{Y}_\infty) = 0$ in (iv), since the system ($\tilde{\mathrm{H}}$) is transformed into system (H) by the symplectic transformation matrix $R^{-1}(t)$. The proof is complete. □

In the next result we examine the limit properties of the comparative index and the dual comparative index in (3.8), when Y is a maximal antiprincipal solution of (H) at ∞ satisfying the properties in Theorem 3.6 and Remark 3.7.

Theorem 3.9. *Assume that* (2.13) *holds with* $\mathcal{I} = [a, \infty)$, *system* (H) *is nonoscillatory at* ∞. *Let* $R(t)$ *be a piecewise continuously differentiable symplectic matrix on* $\mathcal{I}$ *with partition* (3.7) *and assume that system* ($\tilde{\mathrm{H}}$) *with* (3.2) *satisfies* (3.3) *and it is nonoscillatory at* ∞. *Moreover, let* Y *be a maximal antiprincipal solution of* (H) *at* ∞ *such that* $\tilde{Y} := R^{-1}Y$ *is a maximal antiprincipal solution of* ($\tilde{\mathrm{H}}$) *at* ∞. *Then the two limits in* (3.8) *exist if and only if the limit of the rank of* $M(t)$ *for* $t \to \infty$ *exists, i.e., condition* (3.9) *holds. In this case we have the formulas*

$$\lim_{t\to\infty} \mu(Y(t), R(t)E) = \lim_{t\to\infty} \operatorname{ind}\left\{[W(Y(t), R(t)E)]^T X^{-1}(t)\, M(t)\right\}, \tag{3.30}$$

$$\lim_{t\to\infty} \mu^*(Y(t), R(t)E) = \lim_{t\to\infty} \operatorname{ind}\left\{-[W(Y(t), R(t)E)]^T X^{-1}(t)\, M(t)\right\}. \tag{3.31}$$

Proof. Let Y and $\tilde{Y}$ be as in the theorem. Choose $\alpha \in [a, \infty)$ such that $X(t)$ and $\tilde{X}(t)$ are invertible on $[\alpha, \infty)$. Applying the definition of the (dual) comparative index in (2.2) with (2.3) we obtain that

$$\left.\begin{aligned} \mu(Y(t), R(t)E) = \operatorname{ind}\mathcal{P}(t), \quad \mu^*(Y(t), R(t)E) = \operatorname{ind}[-\mathcal{P}(t)], \\ \mathcal{P}(t) := [W(Y(t), R(t)E)]^T X^{-1}(t)\, M(t), \end{aligned}\right\} \quad t \in [\alpha, \infty). \tag{3.32}$$

Since by Theorem 3.6 and (3.27) we know that the Wronskian $W(Y(t), R(t)E)$ is invertible on $[\alpha, \infty)$, it follows that

$$\operatorname{ind}\mathcal{P}(t) + \operatorname{ind}[-\mathcal{P}(t)] = \operatorname{rank}\mathcal{P}(t) \overset{(3.32)}{=} \operatorname{rank} M(t), \quad t \in [\alpha, \infty). \tag{3.33}$$

Due to the continuity of $\mathcal{P}(t)$ and its eigenvalues, the equality in (3.33) implies that the quantities $\operatorname{ind}\mathcal{P}(t)$ and $\operatorname{ind}[-\mathcal{P}(t)]$ are eventually constant (i.e., their limits for $t\to\infty$ exist) if and only if condition (3.9) holds. Consequently, by (3.32) the two limits in (3.8) exist if and only if condition (3.9) is satisfied, and in this case (3.30) and (3.31) hold. □

In the following auxiliary result we study the behavior of the (dual) comparative index in (3.4) for two conjoined bases Y_1 and Y_2 of (H) for large $t\in\mathcal{I}$. We derive an analogue of [16, Lemma 3.2] for the case of $t_0=\infty$.

Lemma 3.10. *Assume that (2.13) holds with $\mathcal{I}=[a,\infty)$, system (H) is nonoscillatory at ∞. Let $R(t)$ be a piecewise continuously differentiable symplectic matrix on $\mathcal{I}$ and assume that system ($\tilde{\mathrm{H}}$) with (3.2) satisfies (3.3) and it is nonoscillatory at ∞. Let Y_1 and Y_2 be conjoined bases of (H) and let $C_{1\infty}$, $C_{2\infty}$, and $C_{\infty,R}$ be the constant representation matrices in (2.23) associated with Y_1, Y_2, and $R\tilde{Y}_\infty$. Then there exists $\alpha\in[a,\infty)$ such that for all $t\in[\alpha,\infty)$*

$$\begin{aligned}\mu(Y_1(t),R(t)E)-\mu(C_{1\infty},C_{\infty,R})+m_{1L}(\infty)-\widetilde{m}_{1L}(\infty)&\\ =\mu(Y_2(t),R(t)E)-\mu(C_{2\infty},C_{\infty,R})+m_{2L}(\infty)-\widetilde{m}_{2L}(\infty),&\qquad(3.34)\end{aligned}$$

$$\mu^*(Y_1(t),R(t)E)-\mu^*(C_{1\infty},C_{\infty,R})=\mu^*(Y_2(t),R(t)E)-\mu^*(C_{2\infty},C_{\infty,R}),\qquad(3.35)$$

where $m_{1L}(\infty)$, $m_{2L}(\infty)$ and $\widetilde{m}_{1L}(\infty)$, $\widetilde{m}_{2L}(\infty)$ are the multiplicities of the focal point at ∞ of the conjoined bases Y_1, Y_2 and $\tilde{Y}_1:=R^{-1}Y_1$, $\tilde{Y}_2:=R^{-1}Y_2$. Equivalently,

$$q(Y_1,t)=q(Y_2,t)\quad\text{and}\quad q^*(Y_1,t)=q^*(Y_2,t)\quad\text{for all } t\in[\alpha,\infty).\qquad(3.36)$$

Proof. We proceed in a similar way to the proof of [16, Lemma 3.2]. Let Y_1 and Y_2 be given conjoined bases of (H). Denote by $\tilde{Y}_1:=R^{-1}Y_1$ and $\tilde{Y}_2:=R^{-1}Y_2$ the corresponding conjoined bases of system ($\tilde{\mathrm{H}}$), and consider the symplectic fundamental matrices $\tilde{Z}_1$ and $\tilde{Z}_2$ of ($\tilde{\mathrm{H}}$) satisfying $\tilde{Z}_1E=\tilde{Y}_1$ and $\tilde{Z}_2E=\tilde{Y}_2$. Fix now $t\in\mathcal{I}$. By formulas (2.8) and (2.10) in Proposition 2.2 (with $V:=R(t)$, $Z_1:=\tilde{Z}_1(t)$, $Z_2:=\tilde{Z}_2(t)$) we obtain the identities

$$\mu(Y_1(t),R(t)E)-\mu(Y_2(t),R(t)E)=\mu(Y_1(t),Y_2(t))-\mu(\tilde{Y}_1(t),\tilde{Y}_2(t)),\qquad(3.37)$$

$$\mu^*(Y_1(t),R(t)E)-\mu^*(Y_2(t),R(t)E)=\mu^*(Y_1(t),Y_2(t))-\mu^*(\tilde{Y}_1(t),\tilde{Y}_2(t)).\qquad(3.38)$$

By Proposition 3.1 we know that the limits of the comparative indices on the right-hand side of (3.37) and (3.38) exist with

$$\lim_{t\to\infty}\mu(Y_1(t),Y_2(t))=\mu(C_{1\infty},C_{2\infty})-m_{1L}(\infty)+m_{2L}(\infty),\qquad(3.39)$$

$$\lim_{t\to\infty}\mu(\tilde{Y}_1(t),\tilde{Y}_2(t))=\mu(\tilde{C}_{1\infty},\tilde{C}_{2\infty})-\widetilde{m}_{1L}(\infty)+\widetilde{m}_{2L}(\infty),\qquad(3.40)$$

$$\lim_{t\to\infty}\mu^*(Y_1(t),Y_2(t))=\mu^*(C_{1\infty},C_{2\infty}),\qquad(3.41)$$

$$\lim_{t\to\infty}\mu^*(\tilde{Y}_1(t),\tilde{Y}_2(t))=\mu^*(\tilde{C}_{1\infty},\tilde{C}_{2\infty}),\qquad(3.42)$$

where $\tilde{C}_{1\infty}$ and $\tilde{C}_{2\infty}$ are the constant matrices in (3.22) associated with $\tilde{Y}_1$ and $\tilde{Y}_2$. This implies that there exists $\alpha \in \mathcal{I}$ such that the (dual) comparative indices in (3.39)–(3.42) are constant on $[\alpha, \infty)$, and hence on this interval we have

$$\begin{aligned}\mu(Y_1(t), R(t)E) - \mu(Y_2(t), R(t)E) \equiv \mu(C_{1\infty}, C_{2\infty}) - \mu(\tilde{C}_{1\infty}, \tilde{C}_{2\infty}) \\ - m_{1L}(\infty) + m_{2L}(\infty) + \widetilde{m}_{1L}(\infty) - \widetilde{m}_{2L}(\infty),\end{aligned} \tag{3.43}$$

$$\mu^*(Y_1(t), R(t)E) - \mu^*(Y_2(t), R(t)E) \equiv \mu^*(C_{1\infty}, C_{2\infty}) - \mu^*(\tilde{C}_{1\infty}, \tilde{C}_{2\infty}). \tag{3.44}$$

Fix now $t \in [\alpha, \infty)$. Applying formulas (2.8) and (2.10) in Proposition 2.2 again (this time with $V := Z_\infty^{-1}(t)\, R(t)\, \tilde{Z}_\infty(t)$, $Z_1 := \tilde{Z}_\infty^{-1}(t)\, \tilde{Z}_1(t)$, $Z_2 := \tilde{Z}_\infty^{-1}(t)\, \tilde{Z}_2(t)$) and using the equalities $Z_\infty^{-1} Y_1 = C_{1\infty}$, $Z_\infty^{-1} Y_2 = C_{2\infty}$, $\tilde{Z}_\infty^{-1} \tilde{Y}_1 = \tilde{C}_{1\infty}$, $\tilde{Z}_\infty^{-1} \tilde{Y}_2 = \tilde{C}_{2\infty}$, and $Z_\infty^{-1} R\, \tilde{Y}_\infty = C_{\infty,R}$ we get

$$\mu(C_{1\infty}, C_{2\infty}) - \mu(\tilde{C}_{1\infty}, \tilde{C}_{2\infty}) = \mu(C_{1\infty}, C_{\infty,R}) - \mu(C_{2\infty}, C_{\infty,R}), \tag{3.45}$$

$$\mu^*(C_{1\infty}, C_{2\infty}) - \mu^*(\tilde{C}_{1\infty}, \tilde{C}_{2\infty}) = \mu^*(C_{1\infty}, C_{\infty,R}) - \mu^*(C_{2\infty}, C_{\infty,R}). \tag{3.46}$$

Upon inserting (3.45) into (3.43) we obtain the equality in (3.34) for all $t \in [\alpha, \infty)$, while using (3.46) in (3.44) yields the equality in (3.35) for all $t \in [\alpha, \infty)$. Finally, the proof of (3.36) follows from the definition of $q(Y, t)$ and $q^*(Y, t)$ in (3.13) and (3.14). □

We are now ready to present the proof of Theorem 3.2.

Proof of Theorem 3.2. Let Y be a conjoined basis of (H) and let R be a symplectic matrix satisfying the assumptions of the theorem. From Remark 3.7 we know that there exists a maximal antiprincipal solution $\bar{Y}$ of (H) at ∞ such that $R^{-1}\bar{Y}$ is a maximal antiprincipal solution of system ($\tilde{\text{H}}$) at ∞. By equations (3.34) and (3.35) in Lemma 3.10 (with $Y_1 := Y$ and $Y_2 := \bar{Y}$), the limits in (3.8) exist if and only if the limits

$$\lim_{t\to\infty} \mu(\bar{Y}(t), R(t)E) \quad \text{and} \quad \lim_{t\to\infty} \mu^*(\bar{Y}(t), R(t)E)$$

both exist. In turn, this is equivalent by Theorem 3.9 (with $Y := \bar{Y}$) to rank $M(t)$ being eventually constant. This proves the first part of Theorem 3.2. Next, under condition (3.9), we will prove (3.11) and (3.12) by deriving their equivalent form (3.17) and (3.18). We know by (3.36) in Lemma 3.10 that the limits of the quantities $q(Y_1, t)$ and $q^*(Y_1, t)$ for $t \to \infty$ exist and their values do not depend on the choice of the conjoined basis Y_1. For the proof of (3.18) we take two choices of $Y_1 := Y$ and $Y_1 := R\, \tilde{Y}_\infty$. In the latter case $C_{1\infty} = C_{\infty,R}$ and hence $\mu^*(C_{1\infty}, C_{\infty,R}) = 0$. Therefore, we calculate

$$\lim_{t\to\infty} q^*(Y, t) = \lim_{t\to\infty} q^*(R\, \tilde{Y}_\infty, t) \overset{(3.14)}{=} \lim_{t\to\infty} \mu^*\big(R(t)\, \tilde{Y}_\infty(t), R(t)E\big).$$

This proves formula (3.18), and hence also formula (3.12). For the proof of formula (3.17) we take two choices of $Y_1 := Y$ and $Y_1 := Y_\infty$. In the latter case $C_{1\infty} = E$ and from properties (2.6) and (2.23) we obtain $\mu(C_{1\infty}, C_{\infty,R}) = \operatorname{rank} W(Y_\infty, R\, \tilde{Y}_\infty)$ and

$$\lim_{t\to\infty} q^*(Y_\infty, t) \overset{(3.14)}{=} \lim_{t\to\infty} \mu^*(Y_\infty(t), R(t)E) - \operatorname{rank} W(Y_\infty, R\tilde{Y}_\infty). \tag{3.47}$$

Therefore, we conclude that

$$\begin{aligned}
\lim_{t\to\infty} q(Y,t) = \lim_{t\to\infty} q(Y_\infty, t) &\overset{(3.15)}{=} \lim_{t\to\infty} \operatorname{rank} M(t) - \lim_{t\to\infty} q^*(Y_\infty, t) - \operatorname{rank} W(Y_\infty, R\tilde{Y}_\infty) \\
&\overset{(3.47)}{=} \lim_{t\to\infty} \operatorname{rank} M(t) - \lim_{t\to\infty} \mu^*(Y_\infty(t), R(t)E) \\
&\overset{(3.20)}{=} \lim_{t\to\infty} \mu^*\big(R^{-1}(t)\, Y_\infty(t), R^{-1}(t)E\big).
\end{aligned}$$

This proves formula (3.17), and hence also formula (3.11). The proof is complete. □

In the final result of this section we apply Theorem 3.2 to the maximal antiprincipal solution of (H) at ∞ used in the statement of Theorem 3.9. In this case we obtain the following explicit formulas for the limits in (3.10). They will be needed in the proof of the main transformation result for the minimal principal solutions at ∞ in Theorem 4.4.

Corollary 3.11. *Assume that (2.13) holds with $\mathcal{I} = [a, \infty)$, system (H) is nonoscillatory at ∞. Let $R(t)$ be a piecewise continuously differentiable symplectic matrix on $\mathcal{I}$ with partition (3.7) such that condition (3.9) holds. In addition, assume that system ($\tilde{\text{H}}$) with (3.2) satisfies (3.3) and it is nonoscillatory at ∞. Moreover, let Y be a maximal antiprincipal solution of (H) at ∞ such that $\tilde{Y} := R^{-1}Y$ is a maximal antiprincipal solution of ($\tilde{\text{H}}$) at ∞. Then the limits in (3.10) exist with*

$$\left.\begin{aligned}
&\lim_{t\to\infty} \mu^*(R^{-1}(t)\, Y_\infty(t), R^{-1}(t)E) \\
&\qquad = \lim_{t\to\infty} \operatorname{ind}\big\{[W(Y(t), R(t)E)]^T X^{-1}(t)\, M(t)\big\} \\
&\qquad\qquad - \operatorname{ind}\big\{-W(\tilde{Y}_\infty, \tilde{Y})\,[W(Y_\infty, Y)]^{-1}\, W(Y_\infty, R\tilde{Y}_\infty)\big\}
\end{aligned}\right\} \tag{3.48}$$

$$\left.\begin{aligned}
&\lim_{t\to\infty} \mu^*(R(t)\, \tilde{Y}_\infty(t), R(t)E) \\
&\qquad = \lim_{t\to\infty} \operatorname{ind}\big\{-[W(Y(t), R(t)E)]^T X^{-1}(t)\, M(t)\big\} \\
&\qquad\qquad - \operatorname{ind}\big\{W(\tilde{Y}_\infty, \tilde{Y})\,[W(Y_\infty, Y)]^{-1}\, W(Y_\infty, R\tilde{Y}_\infty)\big\}.
\end{aligned}\right\} \tag{3.49}$$

Proof. Formula (3.48) follows from equation (3.11) by using (3.30), (3.28), $m_L(\infty) = 0$, and $\widetilde{m}_L(\infty) = 0$. Formula (3.49) follows from the combination of (3.12) with (3.31) and (3.29). □

4. Transformation results under majorant condition

In this section we apply the results from Section 3 to linear Hamiltonian systems (H) and ($\hat{\text{H}}$) satisfying the majorant condition (1.1). The transformed system ($\tilde{\text{H}}$) is now obtained by the special choice of the transformation matrix $R(t) := \hat{Z}(t)$, being the symplectic fundamental matrix of the minorant system ($\hat{\text{H}}$) (as in Propositions 1.2 and 1.3 and in Remark 1.4). In particular, in some results it will be convenient to take the transformation matrix $R(t) := \hat{Z}_\infty(t)$, which is associated with the minimal principal solution $\hat{Y}_\infty$ of ($\hat{\text{H}}$) at ∞ by $\hat{Y}_\infty = \hat{Z}_\infty E$.

First we recall a comparison result for nonoscillatory systems (H) and ($\hat{\text{H}}$) at ∞ under the majorant condition (1.1), as well as the invariance of the nonoscillation at ∞ for system (H) and the transformed system ($\tilde{\text{H}}$). The latter result is based on the generalized reciprocity principle in [16, Theorem 2.2], see also Remark 3.3(i).

Proposition 4.1. *Assume that* (1.1) *and* (1.2) *hold on* $\mathcal{I} = [a, \infty)$ *and that system* (H) *is nonoscillatory at* ∞. *Then*

(i) *system* ($\hat{\text{H}}$) *is nonoscillatory at* ∞, *and*
(ii) *for every symplectic fundamental matrix* $\hat{Z}$ *of* ($\hat{\text{H}}$) *the transformed system* ($\tilde{\text{H}}$) *with the coefficient matrix* $\tilde{\mathcal{H}}(t)$ *given in* (1.14) *is nonoscillatory at* ∞.

Proof. Recall that assumption (1.2) implies under (1.1) the Legendre condition (2.13), and that (1.1) itself implies the Legendre condition (3.3) for the transformed system ($\tilde{\text{H}}$). The nonoscillation of system ($\hat{\text{H}}$) at ∞ follows from the nonoscillation of (H) at ∞ by [35, Theorem 2.6]. Let $\hat{Z}$ be a symplectic fundamental matrix of system ($\hat{\text{H}}$). Then the right upper block $M(t) := (I,\ 0)\, \hat{Z}(t)\, E$ of the transformation matrix $\hat{Z}(t)$ has eventually constant kernel (as system ($\hat{\text{H}}$) is nonoscillatory at ∞), and hence $M(t)$ has also eventually constant rank. Then by Remark 3.3(i) it follows that systems (H) and ($\tilde{\text{H}}$) are simultaneously oscillatory or nonoscillatory at ∞. But since the nonoscillation of (H) at ∞ is a standing assumption of the proposition, it follows that the transformed system ($\tilde{\text{H}}$) is nonoscillatory at ∞ as well. $\square$

Next we present the following auxiliary result.

Lemma 4.2. *Assume that* (1.1) *and* (1.2) *hold on* $\mathcal{I} = [a, \infty)$ *and that system* (H) *is nonoscillatory at* ∞. *Let* $\hat{Z}$ *be a symplectic fundamental matrix of* ($\hat{\text{H}}$). *Let* Y *be a maximal antiprincipal solution of* (H) *at* ∞ *such that* $\tilde{Y} := \hat{Z}^{-1} Y$ *is a maximal antiprincipal solution at* ∞ *of the transformed system* ($\tilde{\text{H}}$) *from Remark 1.4. Assume that* $\alpha \in [a, \infty)$ *is such that* $X(t)$ *is invertible on* $[\alpha, \infty)$, *the conjoined basis* $\hat{Y} := \hat{Z} E$ *of* ($\hat{\text{H}}$) *has constant kernel on* $[\alpha, \infty)$, *and the Wronskian matrix* $W(Y(t), \hat{Y}(t))$ *is invertible on* $[\alpha, \infty)$. *If we set for* $t \in [\alpha, \infty)$

$$\mathcal{P}(t) := [\, W(Y(t), \hat{Y}(t))\,]^T X^{-1}(t)\, \hat{X}(t), \quad \hat{S}(t) := \int_\alpha^t \hat{X}^\dagger(s)\, \hat{B}(s)\, \hat{X}^{\dagger T}(s)\, \mathrm{d}s, \tag{4.1}$$

then the matrix $\mathcal{P}(t)$ *is nondecreasing on* $[\alpha, \infty)$ *and the matrix* $\mathcal{P}^\dagger(t) + \hat{S}(t)$ *is nonincreasing on* $[\alpha, \infty)$.

Proof. The assumptions of the lemma imply that the Legendre conditions (2.13) and (3.3) hold and that systems ($\hat{\text{H}}$) and ($\tilde{\text{H}}$) are nonoscillatory at ∞, by Proposition 4.1. Then there exists $\alpha \in [a, \infty)$ such that $X(t)$ is invertible on $[\alpha, \infty)$ and the conjoined basis $\hat{Y}$ has constant kernel on $[\alpha, \infty)$. Moreover, by Theorem 3.6 (with $R(t) := \hat{Z}(t)$) the point α can be chosen so that the Wronskian matrix $W(Y(t), \hat{Y}(t))$ is invertible on $[\alpha, \infty)$. Then we also have

$$\operatorname{Im} \mathcal{P}(t) = \operatorname{Im} \hat{X}^T(t) = [\operatorname{Ker} \hat{X}(t)]^\perp \ \text{ is constant on } [\alpha, \infty), \tag{4.2}$$

$$\hat{X}(t)\, \hat{X}^\dagger(t)\, \hat{B}(t) = \hat{B}(t) = \hat{B}(t)\, \hat{X}(t)\, \hat{X}^\dagger(t), \quad t \in [\alpha, \infty), \tag{4.3}$$

where (4.3) follows from (2.15) for system ($\hat{\mathrm{H}}$). Observe that $\operatorname{ind}\mathcal{P}(t) = \mu(Y(t), \hat{Y}(t))$ by (3.32). Equation (4.2) yields that $\operatorname{Im}\mathcal{P}(t) \equiv \operatorname{Im}\hat{P}$ is constant on $[\alpha, \infty)$, where

$$\hat{P} := \hat{X}^{\dagger}(t)\,\hat{X}(t) = \mathcal{P}(t)\,\mathcal{P}^{\dagger}(t) = \mathcal{P}^{\dagger}(t)\,\mathcal{P}(t), \quad t \in [\alpha, \infty),$$

is the orthogonal projector onto the constant subspace $\operatorname{Im}\hat{X}^T(t) = [\operatorname{Ker}\hat{X}(t)]^{\perp}$ on $[\alpha, \infty)$. Thus, by [28, Theorem 4.2(iii)], we have $\operatorname{Im}\hat{S}(t) \subseteq \operatorname{Im}\hat{P}$ on $[\alpha, \infty)$, which is equivalent to

$$\mathcal{P}^{\dagger}(t)\,\mathcal{P}(t)\,\hat{S}(t) = \hat{P}\,\hat{S}(t) = \hat{S}(t) = \hat{S}(t)\,\hat{P} = \hat{S}(t)\,\mathcal{P}(t)\,\mathcal{P}^{\dagger}(t), \quad t \in [\alpha, \infty). \tag{4.4}$$

According to [22, Proposition 1.1.3] and (4.3), the derivative of $\mathcal{P}(t)$ satisfies

$$\begin{aligned} \mathcal{P}'(t) &= \mathcal{P}(t)\,\hat{X}^{\dagger}(t)\,\hat{B}(t)\,\hat{X}^{\dagger T}(t)\,\mathcal{P}(t) + V_2^T(t)\,[\mathcal{H}(t) - \hat{\mathcal{H}}(t)]\,V_2(t) \\ &\overset{(4.1)}{=} \mathcal{P}(t)\,\hat{S}'(t)\,\mathcal{P}(t) + V_2^T(t)\,[\mathcal{H}(t) - \hat{\mathcal{H}}(t)]\,V_2(t) \end{aligned} \tag{4.5}$$

for $t \in [\alpha, \infty)$, where $V_2(t) := (I,\ Q(t))^T\,\hat{X}(t)$ and $Q(t) := U(t)\,X^{-1}(t)$. Therefore, $\mathcal{P}'(t) \geq 0$ on $[\alpha, \infty)$ and hence the matrix $\mathcal{P}(t)$ is nondecreasing on the interval $[\alpha, \infty)$. From (4.2) and [20, Theorem 20.8.2] it follows that the matrix $\mathcal{P}^{\dagger}(t)$ is differentiable on $[\alpha, \infty)$ with

$$[\mathcal{P}^{\dagger}(t)]' = -\mathcal{P}^{\dagger}(t)\,\mathcal{P}'(t)\,\mathcal{P}^{\dagger}(t) \overset{(4.5),(4.4)}{=} -\hat{S}'(t) - \mathcal{P}^{\dagger}(t)\,V_2^T(t)\,[\mathcal{H}(t) - \hat{\mathcal{H}}(t)]\,V_2(t)\,\mathcal{P}^{\dagger}(t)$$

on $[\alpha, \infty)$, see also [28, Remark 2.3]. This yields by assumption (1.1) that

$$[\mathcal{P}^{\dagger}(t) + \hat{S}(t)]' = -\mathcal{P}^{\dagger}(t)\,V_2^T(t)\,[\mathcal{H}(t) - \hat{\mathcal{H}}(t)]\,V_2(t)\,\mathcal{P}^{\dagger}(t) \leq 0, \quad t \in [\alpha, \infty),$$

so that the matrix $\mathcal{P}^{\dagger}(t) + \hat{S}(t)$ is nonincreasing on $[\alpha, \infty)$. The proof is complete. □

For the special choice of $\hat{Y} := \hat{Y}_{\infty}$ in Lemma 4.2 we obtain the following important property of the matrix $\mathcal{P}(t)$ in (4.1).

Proposition 4.3. *Assume that (1.1) and (1.2) hold on $\mathcal{I} = [a, \infty)$ and that system (H) is nonoscillatory at ∞. Let $\hat{Z}_{\infty}$ be the symplectic fundamental matrix of ($\hat{\mathrm{H}}$) such that $\hat{Y}_{\infty} = \hat{Z}_{\infty}E$. Let Y be a maximal antiprincipal solution of (H) at ∞ such that $\tilde{Y} := \hat{Z}_{\infty}^{-1}Y$ is a maximal antiprincipal solution at ∞ of the transformed system ($\tilde{\mathrm{H}}$) from Remark 1.4. Then there exists $\alpha \in [a, \infty)$ such that the matrix*

$$\mathcal{P}_{\infty}(t) := [\,W(Y(t), \hat{Y}_{\infty}(t))\,]^T\,X^{-1}(t)\,\hat{X}_{\infty}(t), \quad t \in [\alpha, \infty), \tag{4.6}$$

satisfies

$$\mathcal{P}_{\infty}(t) \leq 0, \quad t \in [\alpha, \infty), \quad \lim_{t\to\infty} \mathcal{P}_{\infty}(t) = 0, \tag{4.7}$$

$$\lim_{t\to\infty} \operatorname{ind}\mathcal{P}_{\infty}(t) = \lim_{t\to\infty} \operatorname{rank}\hat{X}_{\infty}(t) = n - \hat{d}_{\infty}, \tag{4.8}$$

where $\hat{d}_{\infty}$ is the maximal order of abnormality of system ($\hat{\mathrm{H}}$) according to (2.17).

Proof. Let $\alpha \in [a, \infty)$ be a point satisfying the assumptions of Lemma 4.2 (with $\hat{Y} := \hat{Y}_\infty$). In particular, the Wronskian matrix $W(Y(t), \hat{Y}_\infty(t))$ is invertible on $[\alpha, \infty)$. For $t \in [\alpha, \infty)$, let $\hat{S}_\infty(t)$ be the matrix defined in (4.1), which corresponds to $\hat{Y}_\infty$. We denote by $\lambda_j(t)$, $\nu_j(t)$, and $\kappa_j(t)$ for $j \in \{1, \dots, n\}$ and $t \in [\alpha, \infty)$ the eigenvalues of the symmetric and continuous matrices $\mathcal{P}_\infty^\dagger(t)$, $-\hat{S}_\infty(t)$, and $\mathcal{P}_\infty^\dagger(t) + \hat{S}_\infty(t)$, respectively, which are ordered as

$$\lambda_1(t) \le \cdots \le \lambda_n(t), \quad \nu_1(t) \le \cdots \le \nu_n(t), \quad \kappa_1(t) \le \cdots \le \kappa_n(t).$$

Then these eigenvalues are continuous on $[\alpha, \infty)$ as well. By Lemma 4.2 and its proof we know that the matrix $\mathcal{P}_\infty(t)$ has constant rank

$$r := \operatorname{rank} \mathcal{P}_\infty(t) = \operatorname{rank} \hat{X}_\infty(t) \equiv n - \hat{d}_\infty, \quad t \in [\alpha, \infty), \tag{4.9}$$

and that $\mathcal{P}_\infty(t)$ is nondecreasing on this interval. Therefore, the matrix $\mathcal{P}_\infty^\dagger(t)$ is nonincreasing on $[\alpha, \infty)$. Then its eigenvalues $\lambda_j(t)$ are nonincreasing on $[\alpha, \infty)$ and they do not change their sign in this interval. Next, $\hat{S}'_\infty(t) \ge 0$ on $[\alpha, \infty)$ under (1.2), so that the matrix $-\hat{S}_\infty(t) \le 0$ as well as its eigenvalues $\nu_j(t) \le 0$ are nonincreasing on $[\alpha, \infty)$. Moreover, by [28, Theorem 4.2(iii) and Remark 5.3] the set $\operatorname{Im}[-\hat{S}_\infty(t)]$ is eventually constant with

$$\operatorname{rank}[-\hat{S}_\infty(t)] \equiv n - \hat{d}_\infty \overset{(4.9)}{=} r \quad \text{on } [\beta, \infty) \text{ for some } \beta \in [\alpha, \infty).$$

Then the eigenvalues $\nu_j(t)$ satisfy

$$\left.\begin{aligned} \nu_1(t) \le \cdots \le \nu_r(t) < 0, \quad \nu_{r+1}(t) = \cdots = \nu_n(t) = 0, \quad t \in [\beta, \infty), \\ \lim_{t\to\infty} \nu_j(t) = -\infty, \quad j \in \{1, \dots, r\}, \end{aligned}\right\} \tag{4.10}$$

where the last property follows from the fact that $\hat{Y}_\infty$ is a principal solution of ($\hat{\text{H}}$) at ∞, i.e., from $\lim_{t\to\infty} \hat{S}_\infty^\dagger(t) = 0$. Next, by Lemma 4.2 the matrix $\mathcal{P}^\dagger(t) + \hat{S}_\infty(t)$ as well as its eigenvalues $\kappa_j(t)$ are nonincreasing on $[\alpha, \infty)$. Therefore, $\kappa_j(t) \le \kappa_j(\alpha)$ for all $t \in [\alpha, \infty)$ and $j \in \{1, \dots, n\}$. By applying the result about the eigenvalues of a difference of two symmetric matrices in [22, Proposition 3.2.2] (with $Q_1 := \mathcal{P}_\infty^\dagger(t)$ and $Q_2 := -\hat{S}_\infty(t)$) we obtain that $\lambda_j(t) - \nu_j(t) \le \kappa_n(t)$ for every $t \in [\beta, \infty)$ and $j \in \{1, \dots, n\}$. Hence, we get the inequalities

$$\lambda_j(t) \le \nu_j(t) + \kappa_n(t) \le \nu_j(t) + \kappa_n(\alpha), \quad t \in [\beta, \infty), \quad j \in \{1, \dots, n\}. \tag{4.11}$$

By taking the limit in (4.11) and using (4.10) we obtain

$$\lim_{t\to\infty} \lambda_j(t) = -\infty, \quad j \in \{1, \dots, r\}, \tag{4.12}$$

and consequently by (4.9) it follows that

$$\lambda_1(t) \le \cdots \le \lambda_r(t) < 0, \quad \lambda_{r+1}(t) = \cdots = \lambda_n(t) = 0, \quad t \in [\alpha, \infty). \tag{4.13}$$

This shows that $\mathcal{P}_\infty^\dagger(t) \le 0$ on $[\alpha, \infty)$, and hence also $\mathcal{P}_\infty(t) \le 0$ on $[\alpha, \infty)$. Moreover, conditions (4.12) and (4.13) yield that $\lim_{t\to\infty} \mathcal{P}_\infty(t) = \lim_{t\to\infty} [\mathcal{P}_\infty^\dagger(t)]^\dagger = 0$. This completes the proof of (4.7). Finally, by (4.13) we know that $\operatorname{ind}\mathcal{P}_\infty(t) = \operatorname{ind}\mathcal{P}_\infty^\dagger(t) \equiv r$ on $[\alpha, \infty)$, so that $\lim_{t\to\infty} \operatorname{ind}\mathcal{P}_\infty(t) = r$. Taking (4.9) into account then completes the proof of (4.8). □

The following result shows that under natural assumptions the minimal principal solution Y_∞ of (H) at ∞ is transformed into the minimal principal solution $\tilde{Y}_\infty$ of ($\tilde{\text{H}}$) at ∞ and vice versa. We again utilize the transformation matrix $R(t) := \hat{Z}_\infty(t)$ associated with the minimal principal solution $\hat{Y}_\infty$ at ∞ of the minorant system ($\hat{\text{H}}$).

Theorem 4.4. *Assume that (1.1) and (1.2) hold on $\mathcal{I} = [a, \infty)$ and that system (H) is nonoscillatory at ∞. Let $\hat{Z}_\infty$ be the symplectic fundamental matrix of ($\hat{\text{H}}$) such that $\hat{Y}_\infty = \hat{Z}_\infty E$. Then the conjoined basis $\hat{Z}_\infty^{-1} Y_\infty$ is the minimal principal solution of ($\tilde{\text{H}}$) at ∞ and the conjoined basis $\hat{Z}_\infty \tilde{Y}_\infty$ is the minimal principal solution of (H) at ∞. That is, $W(\hat{Z}_\infty^{-1} Y_\infty, \tilde{Y}_\infty) = 0$, $W(Y_\infty, \hat{Z}_\infty \tilde{Y}_\infty) = 0$, and $\hat{Z}_\infty^{-1} Y_\infty = \tilde{Y}_\infty K$ and $\hat{Z}_\infty \tilde{Y}_\infty = Y_\infty K^{-1}$ for some constant invertible $n \times n$ matrix K. Moreover, we have*

$$\lim_{t\to\infty} \mu^*(\hat{Z}_\infty(t)\, \tilde{Y}_\infty(t), \hat{Y}_\infty(t)) = 0, \quad \lim_{t\to\infty} \mu^*(\hat{Z}_\infty^{-1}(t)\, Y_\infty(t), \hat{Z}_\infty^{-1}(t) E) = n - \hat{d}_\infty. \tag{4.14}$$

Proof. By Remark 3.7, for the given transformation matrix $R(t) := \hat{Z}_\infty(t)$ there exists an antiprincipal solution Y of (H) at ∞, which is transformed into a maximal antiprincipal solution $\tilde{Y} := \hat{Z}_\infty^{-1} Y$ of ($\tilde{\text{H}}$) at ∞. Then the constant Wronskian $W(\tilde{Y}_\infty, \tilde{Y})$ is invertible, so that by (3.26) the Wronskian $W(C_{\infty,\hat{Z}_\infty}, C_\infty)$ is invertible as well. Hence, without loss of generality (upon multiplying Y by a constant invertible multiple) we can choose Y such that $W(\tilde{Y}_\infty, \tilde{Y}) = W(C_{\infty,\hat{Z}_\infty}, C_\infty) = I$. Then by Proposition 4.3 the matrix $\mathcal{P}_\infty(t)$ defined in (4.6), which is associated with any such a conjoined basis Y, satisfies $\lim_{t\to\infty} \operatorname{ind}[-\mathcal{P}_\infty(t)] = 0$. Moreover, by (3.49) in Corollary 3.11 (with $R := \hat{Z}_\infty$) we get

$$\begin{aligned} \lim_{t\to\infty} \mu^*(\hat{Z}_\infty(t)\, \tilde{Y}_\infty(t), \hat{Y}_\infty(t)) &= \lim_{t\to\infty} \operatorname{ind}[-\mathcal{P}_\infty(t)] - \operatorname{ind}\big\{[W(Y_\infty, Y)]^{-1}\, W(Y_\infty, \hat{Z}_\infty \tilde{Y}_\infty)\big\} \\ &= -\operatorname{ind}\big\{[W(Y_\infty, Y)]^{-1}\, W(Y_\infty, \hat{Z}_\infty \tilde{Y}_\infty)\big\}. \end{aligned}$$

Since the left-hand side above is nonnegative and the right-hand side is nonpositive, it follows that both sides are zero, that is,

$$\lim_{t\to\infty} \mu^*(\hat{Z}_\infty(t)\, \tilde{Y}_\infty(t), \hat{Y}_\infty(t)) = 0 = \operatorname{ind}\big\{[W(Y_\infty, Y)]^{-1}\, W(Y_\infty, \hat{Z}_\infty \tilde{Y}_\infty)\big\}. \tag{4.15}$$

This shows that the first equality in (4.14) holds. Applying equality (3.25) in Theorem 3.6 (with $R := \hat{Z}_\infty$) together with (4.15) we obtain that

$$\operatorname{ind}\big\{W(Y_\infty, \hat{Z}_\infty \tilde{Y}_\infty)\,(D - D_{\infty,\hat{Z}_\infty})\,[W(Y_\infty, \hat{Z}_\infty \tilde{Y}_\infty)]^T\big\} = 0 \tag{4.16}$$

holds for every symmetric matrix D satisfying

$$\operatorname{Im}\big\{W(Y_\infty, \hat{Z}_\infty \tilde{Y}_\infty)\,(D - D_{\infty,\hat{Z}_\infty})\,[W(Y_\infty, \hat{Z}_\infty \tilde{Y}_\infty)]^T\big\} = \operatorname{Im} W(Y_\infty, \hat{Z}_\infty \tilde{Y}_\infty). \tag{4.17}$$

Then the symmetric matrix $D := D_{\infty,\hat{Z}_\infty} - I$ satisfies (4.17), and hence by (4.16) we have

$$0 = \operatorname{ind}\left\{-W(Y_\infty, \hat{Z}_\infty \tilde{Y}_\infty)\,[W(Y_\infty, \hat{Z}_\infty \tilde{Y}_\infty)]^T\right\} = \operatorname{rank} W(Y_\infty, \hat{Z}_\infty \tilde{Y}_\infty).$$

This implies that $W(\hat{Z}_\infty^{-1} Y_\infty, \tilde{Y}_\infty) = W(Y_\infty, \hat{Z}_\infty \tilde{Y}_\infty) = 0$. Hence, Theorem 3.8 yields that $\hat{Z}_\infty^{-1} Y_\infty$ is the minimal principal solution of ($\tilde{\mathrm{H}}$) at ∞ and $\hat{Z}_\infty \tilde{Y}_\infty$ is the minimal principal solution of (H) at ∞. Finally, we prove the second equality in (4.14) by applying Remark 3.4 with $R(t) := \hat{Z}_\infty(t)$. The limit in (3.18) is equal to zero. Then by (3.15) with $W(Y_\infty, \hat{Z}_\infty \tilde{Y}_\infty) = 0$ and $M(t) := \hat{X}_\infty(t)$ in combination with (3.17) it follows that

$$\lim_{t\to\infty} \mu^*(\hat{Z}_\infty^{-1}(t)\, Y_\infty(t), \hat{Z}_\infty^{-1}(t) E) = \lim_{t\to\infty} \operatorname{rank} \hat{X}_\infty(t) = n - \hat{d}_\infty.$$

The proof is complete. □

Remark 4.5. The result in Theorem 3.6 (respectively in Theorem 3.8) describes the situation when maximal antiprincipal solutions of (H) at ∞ are transformed into maximal antiprincipal solutions of ($\tilde{\mathrm{H}}$) at ∞ under a general symplectic transformation matrix $R(t)$. On the other hand, the result in Theorem 4.4 shows that the minimal principal solution of (H) at ∞ is transformed into the minimal principal solution of ($\tilde{\mathrm{H}}$) at ∞ under the special transformation matrix $R(t) := \hat{Z}_\infty(t)$ and when (1.1) holds. This poses an interesting open problem regarding the general situation, when principal (antiprincipal) solutions of (H) at ∞ would be transformed into some principal (antiprincipal) solutions of ($\tilde{\mathrm{H}}$) at ∞. In the controllable case these results are known in [10, Theorems 1 and 2].

As the final result of this section we derive limit results for the comparative indices in (3.4) with $R(t) := \hat{Z}_\infty(t)$ by applying Theorem 3.2 to this case.

Theorem 4.6. *Assume that (1.1) and (1.2) hold on $\mathcal{I} = [a, \infty)$ and that system (H) is nonoscillatory at ∞. Let $\hat{Z}_\infty$ be the symplectic fundamental matrix of ($\hat{\mathrm{H}}$) such that $\hat{Y}_\infty = \hat{Z}_\infty E$. Then for every conjoined basis Y of (H) the limits*

$$\lim_{t\to\infty} \mu(Y(t), \hat{Y}_\infty(t)) \quad \textit{and} \quad \lim_{t\to\infty} \mu^*(Y(t), \hat{Y}_\infty(t)) \tag{4.18}$$

exist and satisfy

$$\lim_{t\to\infty} \mu(Y(t), \hat{Y}_\infty(t)) = \widetilde{m}_L(\infty) - m_L(\infty) + \widehat{m}_{L\infty}(\infty), \tag{4.19}$$

$$\lim_{t\to\infty} \mu^*(Y(t), \hat{Y}_\infty(t)) = 0, \tag{4.20}$$

where $\widetilde{m}_L(\infty)$ is the multiplicity of a focal point at ∞ of the conjoined basis $\tilde{Y} := \hat{Z}_\infty^{-1} Y$ of the transformed system ($\tilde{\mathrm{H}}$).

Proof. By Proposition 4.1 we know that systems ($\hat{\mathrm{H}}$) and ($\tilde{\mathrm{H}}$) are nonoscillatory at ∞. Then the matrix $M(t) := \hat{X}_\infty(t)$ has eventually constant kernel, and hence condition (3.9) is satisfied in this case. Therefore, by Theorem 3.2 (with $R(t) := \hat{Z}_\infty(t)$) the limits in (4.18) exist. By Theorem 4.4 we have $W(Y_\infty, \hat{Z}_\infty \tilde{Y}_\infty) = 0$, so that by (2.23) the representation matrix $C_{\infty,\hat{Z}_\infty}$ has its first component zero. Then we have $\mu(C_\infty, C_{\infty,\hat{Z}_\infty}) = 0 = \mu^*(C_\infty, C_{\infty,\hat{Z}_\infty})$ by (2.6) and (2.7). Therefore, by using (3.11) and the second equality in (4.14) we get

$$\lim_{t\to\infty} \mu(Y(t), \hat{Y}_\infty(t)) = \widetilde{m}_L(\infty) - m_L(\infty) + n - \hat{d}_\infty = \widetilde{m}_L(\infty) - m_L(\infty) + \widehat{m}_{L\infty}(\infty),$$

where we used that $\widehat{m}_{L\infty}(\infty) = n - \hat{d}_\infty$, see Section 2. This proves (4.19). Similarly, by using (3.12) and the first equality in (4.14) we get (4.20). The proof is complete. □

Remark 4.7. Equation (4.19) is an extension of formula (1.10) in Proposition 1.2 with $\hat{Y} := \hat{Y}_{t_0}$ to the case $t_0 = \infty$. At the same time it is a generalization of formula (3.5) in Proposition 3.1 with $Y^* := Y_\infty$ to two systems (H) and ($\hat{\mathrm{H}}$) satisfying (1.1). Note that equation (4.20) represents an extension to $t_0 = \infty$ of the left continuity of the dual comparative index $\mu^*(Y(t), \hat{Y}_{t_0}(t))$ at the point t_0 in Proposition 1.3.

5. Singular Sturmian comparison theorems

In this section we present the Sturmian comparison theorems for two linear Hamiltonian systems (H) and ($\hat{\mathrm{H}}$) on $\mathcal{I} = [a, \infty)$ satisfying the majorant condition (1.1). In particular, we generalize the comparison theorems in Propositions 1.1, 1.2, and 1.3, as well as the separation theorems in Propositions 2.4 and 2.6 to this case. At the same time we do not assume any controllability condition on systems (H) and ($\hat{\mathrm{H}}$). The above assumptions imply (by Proposition 4.1) that the systems (H), ($\hat{\mathrm{H}}$), and ($\tilde{\mathrm{H}}$) (with $R(t) := \hat{Z}(t)$ being a symplectic fundamental matrix of system ($\hat{\mathrm{H}}$)) are nonoscillatory at ∞. Therefore, their conjoined bases have finitely many left and right focal points in $[a, \infty)$, and hence counting and comparing these numbers over unbounded intervals makes sense. Specifically, we will use the special transformation matrix $R(t) := \hat{Z}_\infty(t)$, as in Section 4.

The main results are formulated by using the notation in (1.5) for the numbers of left and right focal points of conjoined bases of (H), ($\hat{\mathrm{H}}$), ($\tilde{\mathrm{H}}$) in the interval $\mathcal{I}$ including the multiplicities of focal points at ∞. The most general result in this section reads as follows.

Theorem 5.1 *(Singular Sturmian comparison theorem). Assume that* (1.1) *and* (1.2) *hold on* $\mathcal{I} = [a, \infty)$ *and that system* (H) *is nonoscillatory at* ∞. *Let* $\hat{Z}_\infty$ *be the symplectic fundamental matrix of* ($\hat{\mathrm{H}}$) *such that* $\hat{Y}_\infty = \hat{Z}_\infty E$. *Then for every conjoined basis* Y *of* (H) *and for every conjoined basis* $\hat{Y}$ *of* ($\hat{\mathrm{H}}$) *we have*

$$m_L(a, \infty] - \widehat{m}_L(a, \infty] = \widetilde{m}_L(a, \infty] + \mu(\hat{Y}(a), \hat{Y}_\infty(a)) - \mu(Y(a), \hat{Y}_\infty(a)), \tag{5.1}$$

$$m_R[a, \infty) - \widehat{m}_R[a, \infty) = \widetilde{m}_R[a, \infty) + \mu^*(Y(a), \hat{Y}_\infty(a)) - \mu^*(\hat{Y}(a), \hat{Y}_\infty(a)), \tag{5.2}$$

where $\widetilde{m}_L(a, \infty]$ *and* $\widetilde{m}_R[a, \infty)$ *are the numbers of left and right focal points of the conjoined basis* $\tilde{Y} := \hat{Z}_\infty^{-1} Y$ *of* ($\tilde{\mathrm{H}}$) *in the indicated intervals.*

Remark 5.2. (i) When systems (H) and ($\hat{\mathrm{H}}$) coincide (see also Remark 3.5), then for every conjoined basis $\tilde{Y}$ of ($\tilde{\mathrm{H}}$) we have $\widetilde{m}_L(a, \infty] = 0 = \widetilde{m}_R[a, \infty)$. In this case Theorem 5.1 reduces to Proposition 2.4.

(ii) In order to derive a suitable generalization of Proposition 2.4 to two systems (H) and ($\hat{\mathrm{H}}$), it is essential to understand that the differences $m_L(a, \infty] - \widehat{m}_L(a, \infty]$ and $m_R[a, \infty) - \widehat{m}_R[a, \infty)$ in Proposition 2.4 are expressed as (2.25) and (2.27).

(iii) In Propositions 1.2 and 1.3 the transformed system ($\tilde{\mathrm{H}}$) changes with the chosen conjoined basis $\hat{Y}$ of ($\hat{\mathrm{H}}$). The formulation in Theorem 5.1 has the advantage that the transformed system

($\tilde{\mathrm{H}}$) is the same for all conjoined bases $\hat{Y}$ of ($\hat{\mathrm{H}}$), since we use the special transformation matrix $R(t) := \hat{Z}_\infty(t)$, which does not depend on $\hat{Y}$.

The proof of Theorem 5.1 is presented below after the following special result regarding the case of $\hat{Y} := \hat{Y}_\infty$. It is a generalization of the second equality in (2.28) and of the first equality in (2.29) in Proposition 2.6 to two systems (H) and ($\hat{\mathrm{H}}$). At the same time it is extensions of (1.11) and (1.13) to the case of $b = \infty$ with the fundamental matrix $\hat{Z}(t) := \hat{Z}_\infty(t)$.

Theorem 5.3 *(Singular Sturmian comparison theorem). Assume that* (1.1) *and* (1.2) *hold on* $\mathcal{I} = [a, \infty)$ *and that system* (H) *is nonoscillatory at* ∞*. Let* $\hat{Z}_\infty$ *be the symplectic fundamental matrix of* ($\hat{\mathrm{H}}$) *such that* $\hat{Y}_\infty = \hat{Z}_\infty E$*. Then for every conjoined basis* Y *of* (H) *we have*

$$m_L(a, \infty] = \widehat{m}_{L\infty}(a, \infty] + \widetilde{m}_L(a, \infty] - \mu(Y(a), \hat{Y}_\infty(a)), \tag{5.3}$$

$$m_R[a, \infty) = \widehat{m}_{R\infty}[a, \infty) + \widetilde{m}_R[a, \infty) + \mu^*(Y(a), \hat{Y}_\infty(a)), \tag{5.4}$$

where $\widetilde{m}_L(a, \infty]$ *and* $\widetilde{m}_R[a, \infty)$ *are the numbers of left and right focal points of the conjoined basis* $\tilde{Y} := \hat{Z}_\infty^{-1} Y$ *of* ($\tilde{\mathrm{H}}$) *in the indicated intervals.*

Proof. The proof of Theorem 5.3 is based on a combination of Propositions 1.2 and 1.3 (with the special choice $\hat{Y} := \hat{Y}_\infty$) and Theorem 4.6. By taking the limit for $b \to \infty$ in formula (1.11) in Proposition 1.2 we obtain

$$\begin{aligned} m_L(a, \infty) &= \widehat{m}_{L\infty}(a, \infty) + \widetilde{m}_L(a, \infty) + \lim_{b \to \infty} \mu(Y(b), \hat{Y}_\infty(b)) - \mu(Y(a), \hat{Y}_\infty(a)) \\ &\overset{(4.19)}{=} \widehat{m}_{L\infty}(a, \infty] + \widetilde{m}_L(a, \infty] - m_L(\infty) - \mu(Y(a), \hat{Y}_\infty(a)). \end{aligned}$$

Hence, equality (5.3) follows. Similarly, by taking the limit for $b \to \infty$ in formula (1.13) in Proposition 1.3 we get

$$\begin{aligned} m_R[a, \infty) &= \widehat{m}_{R\infty}[a, \infty) + \widetilde{m}_R[a, \infty) + \mu^*(Y(a), \hat{Y}_\infty(a)) - \lim_{b \to \infty} \mu^*(Y(b), \hat{Y}_\infty(b)) \\ &\overset{(4.20)}{=} \widehat{m}_{R\infty}[a, \infty) + \widetilde{m}_R[a, \infty) + \mu^*(Y(a), \hat{Y}_\infty(a)), \end{aligned}$$

which proves equality (5.4). □

The results in Theorem 5.1 now follow from Theorem 5.3 and Proposition 2.6.

Proof of Theorem 5.1. Let Y and $\hat{Y}$ be conjoined bases of (H) and ($\hat{\mathrm{H}}$), respectively. By the second equality in (2.28) in Proposition 2.6 applied to $\hat{Y}$ and $\hat{Y}_\infty$, being conjoined based of system ($\hat{\mathrm{H}}$), we obtain that $\widehat{m}_L(a, \infty] = \widehat{m}_{L\infty}(a, \infty] - \mu(\hat{Y}(a), \hat{Y}_\infty(a))$. Using this equality in (5.3) yields equation (5.1). Similarly, the first equality in (2.29) in Proposition 2.6 applied to $\hat{Y}$ and $\hat{Y}_\infty$ implies that $\widehat{m}_R[a, \infty) = \widehat{m}_{R\infty}[a, \infty) + \mu^*(\hat{Y}(a), \hat{Y}_\infty(a))$, which together with (5.4) yields equation (5.2). The proof is complete. □

The results in Theorem 5.1 (or Theorem 5.3) allow to derive various estimates for the numbers of left and right focal points of conjoined bases of (H) and ($\hat{\mathrm{H}}$). Our first result shows the exact relationship between the numbers of focal points of the (minimal) principal solutions Y_∞, $\hat{Y}_\infty$, $\tilde{Y}_\infty$ and Y_a, $\hat{Y}_a$, $\tilde{Y}_a$ of systems (H), ($\hat{\text{H}}$), ($\tilde{\text{H}}$).

Corollary 5.4. *Assume that* (1.1) *and* (1.2) *hold on* $\mathcal{I} = [a, \infty)$ *and that system* (H) *is nonoscillatory at* ∞*. Then we have*

$$m_{L\infty}(a,\infty] = \widehat{m}_{L\infty}(a,\infty] + \widetilde{m}_{L\infty}(a,\infty] - \mu(Y_\infty(a), \hat{Y}_\infty(a)), \tag{5.5}$$

$$m_{R\infty}[a,\infty) = \widehat{m}_{R\infty}[a,\infty) + \widetilde{m}_{R\infty}[a,\infty) + \mu^*(Y_\infty(a), \hat{Y}_\infty(a)), \tag{5.6}$$

$$m_{La}(a,\infty] = \widehat{m}_{La}(a,\infty] + \widetilde{m}_{La}(a,\infty] + \mu^*(Y_\infty(a), \hat{Y}_\infty(a)), \tag{5.7}$$

$$m_{Ra}[a,\infty) = \widehat{m}_{Ra}[a,\infty) + \widetilde{m}_{Ra}[a,\infty) - \mu(Y_\infty(a), \hat{Y}_\infty(a)), \tag{5.8}$$

Proof. Equalities (5.5) and (5.6) follow from Theorem 5.3 with $Y := Y_\infty$. Indeed, in this case the conjoined basis $\tilde{Y}$ from Theorem 5.3 satisfies $\tilde{Y} = \hat{Z}_\infty^{-1} Y_\infty = \tilde{Y}_\infty$, by Theorem 4.4. Equalities (5.7) and (5.8) then follow directly from (5.6) and (5.5) with the aid of (2.33) for systems (H), ($\hat{\text{H}}$), and ($\tilde{\text{H}}$). □

The following result confirms the intuitively expected fact that, given the same initial conditions, conjoined bases of the majorant system (H) have in general more focal points than conjoined bases of the minorant system ($\hat{\text{H}}$). It is a generalization of the first part of [22, Corollary 7.3.2, pg. 196].

Theorem 5.5 *(Singular Sturmian comparison theorem). Assume that* (1.1) *and* (1.2) *hold on* $\mathcal{I} = [a, \infty)$ *and that system* (H) *is nonoscillatory at* ∞*. Let* $\hat{Z}_\infty$ *be the symplectic fundamental matrix of* ($\hat{\text{H}}$) *such that* $\hat{Y}_\infty = \hat{Z}_\infty E$*. Let* Y *and* $\hat{Y}$ *be conjoined bases of* (H) *and* ($\hat{\text{H}}$)*, respectively, such that* $Y(a) = \hat{Y}(a)\, K$ *for some invertible matrix* K*. Then*

$$m_L(a,\infty] - \widehat{m}_L(a,\infty] = \widetilde{m}_L(a,\infty] \geq 0, \tag{5.9}$$

$$m_R[a,\infty) - \widehat{m}_R[a,\infty) = \widetilde{m}_R[a,\infty) \geq 0, \tag{5.10}$$

where $\widetilde{m}_L(a,\infty]$ *and* $\widetilde{m}_R[a,\infty)$ *are the numbers of left and right focal points of the conjoined basis* $\tilde{Y} := \hat{Z}_\infty^{-1} Y$ *of* ($\tilde{\text{H}}$) *in the indicated intervals.*

Proof. This result follows directly from Theorem 5.1, in which we realize that the equalities $\mu(Y(a), \hat{Y}_\infty(a)) = \mu(\hat{Y}(a), \hat{Y}_\infty(a))$ and $\mu^*(Y(a), \hat{Y}_\infty(a)) = \mu^*(\hat{Y}(a), \hat{Y}_\infty(a))$ hold due to the assumption $Y(a) = \hat{Y}(a)\, K$ and property (2.7) of the comparative index. □

Remark 5.6. The equalities in (5.9) and (5.10) also provide the information about the relation between the numbers of focal points of the conjoined bases Y and $\tilde{Y} := \hat{Z}_\infty^{-1} Y$. Namely, we have $m_L(a,\infty] \geq \widetilde{m}_L(a,\infty]$ and $m_R[a,\infty) \geq \widetilde{m}_R[a,\infty)$.

Next we compare the numbers of focal points of the (minimal) principal solutions Y_a, $\hat{Y}_a$, $\tilde{Y}_a$ and Y_∞, $\hat{Y}_\infty$, $\tilde{Y}_\infty$. We also provide universal lower and upper bounds for the numbers of focal points of conjoined bases of (H) and ($\hat{\text{H}}$).

Corollary 5.7. *Assume that (1.1) and (1.2) hold on $\mathcal{I} = [a, \infty)$ and that system (H) is nonoscillatory at ∞. Then for every conjoined basis Y of (H) the estimates in (2.30) and (2.31) hold, where the lower and upper bounds satisfy*

$$\widehat{m}_{La}(a,\infty] + \widetilde{m}_{La}(a,\infty] \le m_{La}(a,\infty], \quad m_{L\infty}(a,\infty] \le \widehat{m}_{L\infty}(a,\infty] + \widetilde{m}_{L\infty}(a,\infty], \tag{5.11}$$

$$\widehat{m}_{R\infty}[a,\infty) + \widetilde{m}_{R\infty}[a,\infty) \le m_{R\infty}[a,\infty), \quad m_{Ra}[a,\infty) \le \widehat{m}_{Ra}[a,\infty) + \widetilde{m}_{Ra}[a,\infty). \tag{5.12}$$

Moreover, for every conjoined basis $\hat{Y}$ of ($\hat{\mathrm{H}}$) we have the estimates

$$\widehat{m}_{La}(a,\infty] \le \widehat{m}_{L}(a,\infty] \le \widehat{m}_{L\infty}(a,\infty] \le m_{L\infty}(a,\infty], \tag{5.13}$$

$$\widehat{m}_{R\infty}[a,\infty) \le \widehat{m}_{R}[a,\infty) \le \widehat{m}_{Ra}[a,\infty) \le m_{Ra}[a,\infty). \tag{5.14}$$

Proof. Let Y be a conjoined basis of (H). Then (2.30) and (2.31) are guaranteed by Proposition 2.6. The inequalities in (5.11) follow from (5.7) and (5.5), and similarly the inequalities in (5.12) follow from (5.6) and (5.8). Next, the first two inequalities in (5.13)–(5.14) are guaranteed by (2.30)–(2.31) in Proposition 2.6 applied to system ($\hat{\mathrm{H}}$). For the third inequality in (5.13) we consider equation (5.9) with $\hat{Y} := \hat{Y}_\infty$ and with Y to be the conjoined basis of (H) satisfying the initial condition $Y(a) = \hat{Y}_\infty(a)$. Then $m_L(a,\infty] \ge \widehat{m}_{L\infty}(a,\infty]$ by Theorem 5.5. At the same time, for the conjoined bases Y and Y_∞ of (H) we have the inequality $m_L(a,\infty] \le m_{L\infty}(a,\infty]$, by (2.30). Therefore, $\widehat{m}_{L\infty}(a,\infty] \le m_{L\infty}(a,\infty]$ holds, which completes the proof of (5.13). Finally, since $Y_a(a) = E = \hat{Y}_a(a)$ holds, then the third inequality in (5.14) follows from Theorem 5.5 (with $Y := Y_a$ and $\hat{Y} := \hat{Y}_a$). The proof is complete. □

The estimates in Corollary 5.7 show that the proper generalization of Proposition 1.1 to uncontrollable systems should be done through the right focal points. At the same time Corollary 5.7 provides an extension of Proposition 1.1 to the left focal points.

Remark 5.8. The results of this paper show the importance of the transformed system ($\tilde{\mathrm{H}}$) for comparing the numbers of focal points of conjoined bases of systems (H) and ($\hat{\mathrm{H}}$). Therefore, the transformed system ($\tilde{\mathrm{H}}$) can be considered as a quantitative measure of the majorant condition (1.1). For example, the optimal upper bounds $m_{L\infty}(a,\infty] = m_{Ra}[a,\infty)$ and $\widehat{m}_{L\infty}(a,\infty] = \widehat{m}_{Ra}[a,\infty)$ for the numbers of left and right focal points of conjoined bases Y and $\hat{Y}$ of (H) and ($\hat{\mathrm{H}}$) satisfy the estimates

$$0 \le m_{L\infty}(a,\infty] - \widehat{m}_{L\infty}(a,\infty] \le \widetilde{m}_{L\infty}(a,\infty],$$

$$0 \le m_{Ra}[a,\infty) - \widehat{m}_{Ra}[a,\infty) \le \widetilde{m}_{Ra}[a,\infty).$$

We believe that further investigation of the properties of the transformed system ($\tilde{\mathrm{H}}$) will lead to better explanation of the role of condition (1.1) in the Sturmian theory of these systems. Another open problem is to understand how the maximal orders of abnormality d_∞, $\hat{d}_\infty$, $\tilde{d}_\infty$ of systems (H), ($\hat{\mathrm{H}}$), ($\tilde{\mathrm{H}}$) affect the transformation rules for principal and antiprincipal solutions of (H) at ∞. We will address these issues in our subsequent work.

6. Further Sturmian comparison theorems on unbounded intervals

The results of this paper extend in analogous way to the unbounded intervals of the form $(-\infty, b]$, where for measuring the numbers of focal points we use the corresponding (minimal) principal solutions $Y_{-\infty}$, $\hat{Y}_{-\infty}$, $\tilde{Y}_{-\infty}$ and Y_b, $\hat{Y}_b$, $\tilde{Y}_b$. For the definition of a principal solution of (H) at $-\infty$ we refer to [33, Section 2]. The corresponding Sturmian separation theorems on the intervals of the form $(-\infty, b]$ and the limit results for the comparative index at $-\infty$ for conjoined bases of one system (H) are discussed in [33, Remarks 5.16 and 6.5]. For completeness and future reference we provide the statements for two systems (H) and ($\hat{\text{H}}$) below.

Consider the linear Hamiltonian systems (H) and ($\hat{\text{H}}$), as well as the transformed system ($\tilde{\text{H}}$), on the unbounded interval $\mathcal{I} = (-\infty, b]$. We assume that they satisfy majorant condition (1.1) and the Legendre condition (1.2) on this interval, which in turn implies the validity of the Legendre conditions (2.13) and (3.3) on $(-\infty, b]$. If Y is a conjoined basis of (H) with constant kernel on $(-\infty, \beta]$ for some $\beta \in (-\infty, b]$ with $d(-\infty, \beta] = d_{-\infty}$, then analogously to (2.19) we define the symmetric matrix

$$T_{\beta,-\infty} := \lim_{t\to-\infty} \left(\int_{\beta}^{t} X^{\dagger}(s)\, B(s)\, X^{\dagger T}(s)\, \mathrm{d}s \right)^{\dagger}, \quad 0 \le \operatorname{rank} T_{\beta,-\infty} \le n - d_{-\infty}, \tag{6.1}$$

compare with [32, Section 5]. Note that $T_{\beta,-\infty} \le 0$. Following [33, Remark 3.6] we then define the multiplicity of the (right) proper focal point of Y at $-\infty$ by

$$m_R(-\infty) := n - d_{-\infty} - \operatorname{rank} T_{\beta,-\infty}, \quad 0 \le m_R(-\infty) \le n - d_{-\infty}. \tag{6.2}$$

In this case $m_R(-\infty) = 0$ if and only if Y is an antiprincipal solution of (H) at $-\infty$ (i.e., $\operatorname{rank} T_{\beta,-\infty} = n - d_{-\infty}$), while $m_R(-\infty) = n - d_{-\infty}$ if and only if Y is a principal solution of (H) at $-\infty$ (i.e., $T_{\beta,-\infty} = 0$). Furthermore, as in (2.21) we have the formula

$$m_R(-\infty) = \lim_{t\to-\infty} \operatorname{rank} X(t) - \operatorname{rank} W(Y_{-\infty}, Y), \tag{6.3}$$

where $Y_{-\infty}$ is the minimal principal solution of (H) at $-\infty$. We begin with an analogue of Proposition 4.1.

Proposition 6.1. *Assume that (1.1) and (1.2) hold on $\mathcal{I} = (-\infty, b]$ and that system (H) is nonoscillatory at $-\infty$. Then*

(i) *system ($\hat{\text{H}}$) is nonoscillatory at $-\infty$, and*
(ii) *for every symplectic fundamental matrix $\hat{Z}$ of ($\hat{\text{H}}$) the transformed system ($\tilde{\text{H}}$) with the coefficient matrix $\tilde{\mathcal{H}}(t)$ given in (1.14) is nonoscillatory at $-\infty$.*

Next we present an analogue of Theorem 4.6. Observe that formula (6.5) below is an extension of (1.12) to the case of $t_0 = -\infty$.

Theorem 6.2. *Assume that (1.1) and (1.2) hold on the interval $\mathcal{I} = (-\infty, b]$ and that system (H) is nonoscillatory at $-\infty$. Let $\hat{Z}_{-\infty}$ be the symplectic fundamental matrix of ($\hat{\text{H}}$) such that $\hat{Y}_{-\infty} = \hat{Z}_{-\infty} E$. Then for every conjoined basis Y of (H) the limits*

$$\lim_{t\to-\infty} \mu(Y(t), \hat{Y}_{-\infty}(t)) \quad \textit{and} \quad \lim_{t\to-\infty} \mu^*(Y(t), \hat{Y}_{-\infty}(t))$$

exist and satisfy

$$\lim_{t\to-\infty} \mu(Y(t), \hat{Y}_{-\infty}(t)) = 0, \tag{6.4}$$

$$\lim_{t\to-\infty} \mu^*(Y(t), \hat{Y}_{-\infty}(t)) = \widetilde{m}_R(-\infty) - m_R(-\infty) + \widehat{m}_{R-\infty}(-\infty), \tag{6.5}$$

where $\widetilde{m}_R(-\infty)$ is the multiplicity of a focal point at $-\infty$ of the conjoined basis $\tilde{Y} := \hat{Z}_{-\infty}^{-1} Y$ of the transformed system ($\tilde{\text{H}}$).

Next we present a transformation result for the minimal principal solutions at $-\infty$ and for the maximal antiprincipal solutions at $-\infty$. It is an analogue of Theorems 3.8 and 4.4.

Theorem 6.3. *Assume that* (1.1) *and* (1.2) *hold on the interval $\mathcal{I} = (-\infty, b]$ and that system* (H) *is nonoscillatory at $-\infty$. Let $\hat{Z}_{-\infty}$ be the symplectic fundamental matrix of* ($\hat{\text{H}}$) *such that $\hat{Y}_{-\infty} = \hat{Z}_{-\infty} E$. Then the following statements hold.*

- (i) *The conjoined basis $\hat{Z}_{-\infty}^{-1} Y_{-\infty}$ is the minimal principal solution of* ($\tilde{\text{H}}$) *at $-\infty$ and the conjoined basis $\hat{Z}_{-\infty} \tilde{Y}_{-\infty}$ is the minimal principal solution of* (H) *at $-\infty$.*
- (ii) *Every maximal antiprincipal solution of* (H) *at $-\infty$ is transformed into a maximal antiprincipal solution of* ($\tilde{\text{H}}$) *at $-\infty$ under the transformation $\tilde{Y} = \hat{Z}_{-\infty}^{-1}(t)\, Y$.*
- (iii) *Every maximal antiprincipal solution of* ($\tilde{\text{H}}$) *at $-\infty$ is a transformation of some maximal antiprincipal solution of* (H) *at $-\infty$ under $\tilde{Y} = \hat{Z}_{-\infty}^{-1}(t)\, Y$.*

The results in Theorems 5.1 and 5.3 have the following counterpart. Observe that formulas (6.8) and (6.9) below are extensions of (1.11) and (1.13) to the case of $a = -\infty$ with the fundamental matrix $\hat{Z}(t) := \hat{Z}_{-\infty}(t)$.

Theorem 6.4 *(Singular Sturmian comparison theorem). Assume that* (1.1) *and* (1.2) *hold on $\mathcal{I} = (-\infty, b]$ and that system* (H) *is nonoscillatory at $-\infty$. Let $\hat{Z}_{-\infty}$ be the symplectic fundamental matrix of* ($\hat{\text{H}}$) *such that $\hat{Y}_{-\infty} = \hat{Z}_{-\infty} E$. Then for every conjoined basis Y of* (H) *and for every conjoined basis $\hat{Y}$ of* ($\hat{\text{H}}$) *we have*

$$m_L(-\infty, b] - \widehat{m}_L(-\infty, b] = \widetilde{m}_L(-\infty, b] + \mu(Y(b), \hat{Y}_{-\infty}(b)) - \mu(\hat{Y}(b), \hat{Y}_{-\infty}(b)), \tag{6.6}$$

$$m_R[-\infty, b) - \widehat{m}_R[-\infty, b) = \widetilde{m}_R[-\infty, b) + \mu^*(\hat{Y}(b), \hat{Y}_{-\infty}(b)) - \mu^*(Y(b), \hat{Y}_{-\infty}(b)), \tag{6.7}$$

where $\widetilde{m}_L(-\infty, b]$ and $\widetilde{m}_R[-\infty, b)$ are the numbers of left and right focal points of the conjoined basis $\tilde{Y} := \hat{Z}_{-\infty}^{-1} Y$ of ($\tilde{\text{H}}$) *in the indicated intervals. In particular,*

$$m_L(-\infty, b] = \widehat{m}_{L-\infty}(-\infty, b] + \widetilde{m}_L(-\infty, b] + \mu(Y(b), \hat{Y}_{-\infty}(b)), \tag{6.8}$$

$$m_R[-\infty, b) = \widehat{m}_{R-\infty}[-\infty, b) + \widetilde{m}_R[-\infty, b) - \mu^*(Y(b), \hat{Y}_{-\infty}(b)). \tag{6.9}$$

Finally, the numbers of focal points of the (minimal) principal solutions $Y_{-\infty}$, $\hat{Y}_{-\infty}$, $\tilde{Y}_{-\infty}$ and Y_b, $\hat{Y}_b$, $\tilde{Y}_b$ are related as follows. It is a counterpart of Corollaries 5.4 and 5.7.

Corollary 6.5. *Assume that* (1.1) *and* (1.2) *hold on* $\mathcal{I} = (-\infty, b]$ *and that system* (H) *is nonoscillatory at* $-\infty$. *Then we have the equalities*

$$m_{L-\infty}(-\infty, b] = \widehat{m}_{L-\infty}(-\infty, b] + \widetilde{m}_{L-\infty}(-\infty, b] + \mu(Y_{-\infty}(b), \hat{Y}_{-\infty}(b)), \tag{6.10}$$

$$m_{R-\infty}[-\infty, b) = \widehat{m}_{R-\infty}[-\infty, b) + \widetilde{m}_{R-\infty}[-\infty, b) - \mu^*(Y_{-\infty}(b), \hat{Y}_{-\infty}(b)), \tag{6.11}$$

$$m_{Lb}(-\infty, b] = \widehat{m}_{Lb}(-\infty, b] + \widetilde{m}_{Lb}(-\infty, b] - \mu^*(Y_{-\infty}(b), \hat{Y}_{-\infty}(b)), \tag{6.12}$$

$$m_{Rb}[-\infty, b) = \widehat{m}_{Rb}[-\infty, b) + \widetilde{m}_{Rb}[-\infty, b) + \mu(Y_{-\infty}(b), \hat{Y}_{-\infty}(b)). \tag{6.13}$$

Moreover, we have the estimates

$$\widehat{m}_{L-\infty}(-\infty, b] + \widetilde{m}_{L-\infty}(-\infty, b] \leq m_{L-\infty}(-\infty, b], \tag{6.14}$$

$$\widehat{m}_{Lb}(-\infty, b] \leq m_{Lb}(-\infty, b] \leq \widehat{m}_{Lb}(-\infty, b] + \widetilde{m}_{Lb}(-\infty, b], \tag{6.15}$$

$$\widehat{m}_{R-\infty}[-\infty, b) \leq m_{R-\infty}[-\infty, b) \leq \widehat{m}_{R-\infty}[-\infty, b) + \widetilde{m}_{R-\infty}[-\infty, b), \tag{6.16}$$

$$\widehat{m}_{Rb}[-\infty, b) + \widetilde{m}_{Rb}[-\infty, b) \leq m_{Rb}[-\infty, b). \tag{6.17}$$

Remark 6.6. If assumptions (1.1) and (1.2) hold on $\mathcal{I} = (-\infty, \infty)$ and system (H) is nonoscillatory at $\pm\infty$, then for every conjoined basis Y of (H) we have the estimates

$$m_L(-\infty, \infty) \geq m_{L-\infty}(-\infty, \infty) \geq \widehat{m}_{L-\infty}(-\infty, \infty) + \widetilde{m}_{L-\infty}(-\infty, \infty), \tag{6.18}$$

$$m_R(-\infty, \infty) \geq m_{R\infty}(-\infty, \infty) \geq \widehat{m}_{R\infty}(-\infty, \infty) + \widetilde{m}_{R\infty}(-\infty, \infty). \tag{6.19}$$

The inequalities in (6.18) follow from (6.10) with $b \to \infty$ by dropping the last term with the comparative index, while the inequalities in (6.19) follow from (5.6) with $a \to -\infty$ by dropping the last term with the dual comparative index. These lower bounds for the numbers of left and right focal points of Y in $(-\infty, \infty)$ improve the lower bound $\widehat{m}_\infty(-\infty, \infty)$ obtained in (1.9) of Proposition 1.1 for $a \to -\infty$.

Remark 6.7. Unlike in [33, Remark 8.1] for the Sturmian separation theorems, the results in Section 5 on $\mathcal{I} = [a, \infty)$ together with the results in this section on $\mathcal{I} = (-\infty, b]$ do not combine in general to Sturmian comparison theorems on the entire interval $\mathcal{I} = (-\infty, \infty)$. The main reason is that we employ two different transformation matrices $R(t) = \hat{Z}_{\pm\infty}(t)$ in neighborhoods of $\pm\infty$, which yield two different transformation systems ($\tilde{\text{H}}$). Therefore, the question of the validity of the Sturmian comparison theorems for systems (H) and ($\hat{\text{H}}$) on the unbounded interval $\mathcal{I} = (-\infty, \infty)$ remains an open problem when the minimal principal solutions $\hat{Y}_\infty$ and $\hat{Y}_{-\infty}$ of ($\hat{\text{H}}$) at $\pm\infty$ differ, meaning that $\hat{Y}_{-\infty}$ is not a constant nonsingular multiple of $\hat{Y}_\infty$. In other words, the results presented in Sections 4 and 5 (with $a = -\infty$) and in Section 6 (with $b = \infty$) remain valid under the additional assumption that $\hat{Y}_\infty = \hat{Y}_{-\infty}$ (i.e., $\hat{Z}_\infty = \hat{Z}_{-\infty}$).

Following the discussion in Remark 6.7 we present below the extensions of Theorem 6.4 and Corollary 6.5 to the case of $\mathcal{I} = (-\infty, \infty)$ under the additional assumption $\hat{Y}_\infty = \hat{Y}_{-\infty}$.

Theorem 6.8 *(Singular Sturmian comparison theorem). Assume that* (1.1) *and* (1.2) *hold on* $\mathcal{I} = (-\infty, \infty)$*, system* (H) *is nonoscillatory at* $\pm\infty$*, and that* $\hat{Y}_{-\infty} = \hat{Y}_{\infty}$*. Let* $\hat{Z}_{\infty}$ *be the symplectic fundamental matrix of* ($\hat{\mathrm{H}}$) *such that* $\hat{Y}_{\infty} = \hat{Z}_{\infty} E$*. Then for every conjoined basis* Y *of* (H) *and for every conjoined basis* $\hat{Y}$ *of* ($\hat{\mathrm{H}}$) *we have*

$$m_L(-\infty, \infty] - \widehat{m}_L(-\infty, \infty] = \widetilde{m}_L(-\infty, \infty], \tag{6.20}$$

$$m_R[-\infty, \infty) - \widehat{m}_R[-\infty, \infty) = \widetilde{m}_R[-\infty, \infty), \tag{6.21}$$

where $\widetilde{m}_L(-\infty, \infty]$ *and* $\widetilde{m}_R[-\infty, \infty)$ *are the numbers of left and right focal points of the conjoined basis* $\tilde{Y} := \hat{Z}_{\infty}^{-1} Y$ *of* ($\tilde{\mathrm{H}}$) *in the indicated intervals. In particular,*

$$m_{L-\infty}(-\infty, \infty] = \widehat{m}_{L-\infty}(-\infty, \infty] + \widetilde{m}_{L-\infty}(-\infty, \infty], \tag{6.22}$$

$$m_{R-\infty}[-\infty, \infty) = \widehat{m}_{R-\infty}[-\infty, \infty) + \widetilde{m}_{R-\infty}[-\infty, \infty), \tag{6.23}$$

$$m_{L\infty}(-\infty, \infty] = \widehat{m}_{L\infty}(-\infty, \infty] + \widetilde{m}_{L\infty}(-\infty, \infty], \tag{6.24}$$

$$m_{R\infty}[-\infty, \infty) = \widehat{m}_{R\infty}[-\infty, \infty) + \widetilde{m}_{R\infty}[-\infty, \infty), \tag{6.25}$$

and

$$m_{L-\infty}(-\infty, \infty] \geq \widehat{m}_{L-\infty}(-\infty, \infty], \quad m_{R\infty}[-\infty, \infty) \geq \widehat{m}_{R\infty}[-\infty, \infty). \tag{6.26}$$

Proof. Let Y and $\hat{Y}$ be conjoined bases of (H) and ($\hat{\mathrm{H}}$), respectively. Under the given assumptions we fix an arbitrary point $a \in \mathbb{R}$. Then Theorem 5.1 yields that (5.1) and (5.2) hold, where $\tilde{Y} := \hat{Z}_{\infty}^{-1} Y$. Next, from Theorem 6.4 (with $b := a$) we obtain

$$m_L(-\infty, a] - \widehat{m}_L(-\infty, a] = \widetilde{m}_L(-\infty, a] + \mu(Y(a), \hat{Y}_{-\infty}(a)) - \mu(\hat{Y}(a), \hat{Y}_{-\infty}(a)), \tag{6.27}$$

$$m_R[-\infty, a) - \widehat{m}_R[-\infty, a) = \widetilde{m}_R[-\infty, a) + \mu^*(\hat{Y}(a), \hat{Y}_{-\infty}(a)) - \mu^*(Y(a), \hat{Y}_{-\infty}(a)), \tag{6.28}$$

where $\tilde{Y} := \hat{Z}_{-\infty}^{-1} Y$. Since we now assume that $\hat{Y}_{-\infty} = \hat{Y}_{\infty}$, then we may take $\hat{Z}_{-\infty} = \hat{Z}_{\infty}$, so that the conjoined basis $\tilde{Y}$ in (5.1) and (5.2) coincides with the conjoined basis $\tilde{Y}$ in (6.27) and (6.28). Upon adding equation (5.1) with (6.27), and equation (5.2) with (6.28), and using that $\hat{Y}_{-\infty} = \hat{Y}_{\infty}$ holds, we obtain the equalities in (6.20) and (6.21). In particular, for $Y := Y_{-\infty}$ and $\hat{Y} := \hat{Y}_{-\infty}$ we have $\tilde{Y} = \hat{Z}_{-\infty}^{-1} Y_{-\infty} = \tilde{Y}_{-\infty}$ by Theorem 6.3, so that equality (6.22) follows from (6.20), and equality (6.23) follows from (6.21). Similarly, equalities (6.24) and (6.25) follow from (6.20) and (6.21) with $Y := Y_{\infty}$ and $\hat{Y} := \hat{Y}_{\infty}$ (noting that $\tilde{Y} = \hat{Z}_{\infty}^{-1} Y_{\infty} = \tilde{Y}_{\infty}$ by Theorem 4.4). Finally, the estimates in (6.26) are obtained from (6.22) and (6.25) by dropping the last term in the sum on the right-hand side. □

Remark 6.9. As we mentioned in Section 1, the results of this paper are new even for the case of "weakly disconjugate" linear Hamiltonian systems (H) and ($\hat{\mathrm{H}}$) at ∞, resp. at $-\infty$, i.e., for systems with

$$d_{\infty} = 0 = \hat{d}_{\infty}, \quad \text{resp.} \quad d_{-\infty} = 0 = \hat{d}_{-\infty}. \tag{6.29}$$

In particular, the results are new for completely controllable systems (H) and ($\hat{\text{H}}$), which automatically satisfy condition (6.29). Among the latter ones we mention the second order Sturm–Liouville differential equations. We will present consequences of our new theory for this special case in a separate study.

Appendix A. Auxiliary results about normalized conjoined bases

In this section we present a completion of some known results from matrix analysis related to normalized conjoined bases. The basic problem is to find for a given constant $2n \times n$ matrix Y satisfying $Y^T \mathcal{J} Y = 0$ and $\operatorname{rank} Y = n$ (called for simplicity also a conjoined basis) another constant $2n \times n$ matrix $\hat{Y}$ satisfying the same properties such that their Wronskian matrix $W(Y, \hat{Y}) = I$. In this case we say that the matrices Y and $\hat{Y}$ are *normalized*. A classification of all such conjoined bases $\hat{Y}$ with $W(Y, \hat{Y}) = I$ is derived in [22, Corollary 3.3.9], where it is also stated that the conjoined basis $\hat{Y}$ may be chosen so that the first component $\hat{X} := (I,\ 0)\,\hat{Y}$ is invertible, see also [22, Proposition 4.1.1]. In this section we complete this result by deriving a classification of all such conjoined bases $\hat{Y}$ with $\hat{X}$ invertible. This result is then used in the proof of Theorem 3.6, which is the main tool for the transformation theory of principal and antiprincipal solutions at ∞ (Theorems 3.8 and 4.4).

In accordance with (2.14) we split Y and $\hat{Y}$ into their $n \times n$ components $Y = (X^T, U^T)^T$ and $\hat{Y} = (\hat{X}^T, \hat{U}^T)^T$. Then $W(Y, \hat{Y}) = I$ if and only if $Y\hat{Y}^T - \hat{Y}Y^T = \mathcal{J}$, that is,

$$X\hat{U}^T - \hat{X}U^T = I, \quad X\hat{X}^T \text{ and } U\hat{U}^T \text{ are symmetric,} \tag{A.1}$$

see [22, Proposition 1.1.5]. Moreover, we define the $n \times n$ matrices

$$F := Y^T Y > 0, \quad G := F^{-1/2} > 0, \quad H := GX^T (GX^T)^\dagger, \tag{A.2}$$

$$D_Y := GHGU^T (GX^T)^\dagger G. \tag{A.3}$$

We note that the matrix H is the orthogonal projector onto the subspace $\operatorname{Im}(GX^T)$.

Lemma A.1. *Let Y be a conjoined basis with F, G, H, and D_Y defined in* (A.2) *and* (A.3)*. Then the matrix D_Y is symmetric and satisfies $XD_YX^T = UF^{-1}X^T$.*

Proof. First we note that the matrix $\hat{Y} := -\mathcal{J}YF^{-1}$ is a conjoined basis satisfying $W(Y, \hat{Y}) = I$. Then by (A.1) the matrix $XF^{-1}U^T = -X\hat{X}^T$ is symmetric. The symmetry of the matrix D_Y then follows by

$$D_Y = [(GX^T)^\dagger G]^T XF^{-1}U^T [(GX^T)^\dagger G].$$

Moreover, since $XGH = XG$ and $G^2 = F^{-1}$, it follows that

$$XD_YX^T = XF^{-1}U^T (GX^T)^\dagger GX^T = UGGX^T (GX^T)^\dagger GX^T = UGGX^T = UF^{-1}X^T,$$

which completes the proof. $\square$

Next we present the main result of this section.

Theorem A.2. *Let Y be a conjoined basis with F and D_Y defined in (A.2) and (A.3). Then a matrix $\hat{Y}$ is a conjoined basis with $W(Y, \hat{Y}) = I$ and with the matrix $\hat{X}$ invertible if and only if $\hat{Y} = -\mathcal{J} Y F^{-1} + Y D$, where D is a symmetric $n \times n$ matrix satisfying*

$$\operatorname{Im}[\, X (D - D_Y)\, X^T\,] = \operatorname{Im} X. \tag{A.4}$$

In this case we have

$$\operatorname{ind}[\, X (D - D_Y)\, X^T\,] = \operatorname{ind}(\hat{X}^{-1} X). \tag{A.5}$$

Remark A.3. (i) Condition (A.4) is equivalent with the inclusion $\operatorname{Im} X \subseteq \operatorname{Im}[X (D - D_Y)\, X^T]$, or with $\operatorname{Ker}[X (D - D_Y)\, X^T] \subseteq \operatorname{Ker} X^T$, since the opposite inclusions hold trivially.

(ii) We note that for a given conjoined basis Y there always exists a symmetric matrix D satisfying equality (A.4), e.g. $D := D_Y + X^\dagger X^{\dagger T}$, compare also with [22, Corollary 3.3.9]. Hence, there always exists a conjoined basis $\hat{Y}$ with $W(Y, \hat{Y}) = I$ and $\hat{X}$ invertible.

Proof of Theorem A.2. By [22, Corollary 3.3.9], every conjoined basis $\hat{Y}$ with $W(Y, \hat{Y}) = I$ is of the form $\hat{Y} = -\mathcal{J} Y F^{-1} + Y D$ with a symmetric $n \times n$ matrix D. In particular, the corresponding matrix $\hat{X}$ has the form $\hat{X} = -U F^{-1} + X D$. For a given D we will show that

$$\operatorname{Ker} \hat{X} = X^T \operatorname{Ker}[\, X (D - D_Y)\, X^T\,]. \tag{A.6}$$

Let $v \in \operatorname{Ker} \hat{X}$. Since $W(Y, \hat{Y}) = I$, we have the formula $X^T \hat{U} - U^T \hat{X} = I$ and hence $v = (X^T \hat{U} - U^T \hat{X})\, v = X^T w$ with $w := \hat{U} v$. Moreover, by Lemma A.1,

$$X (D - D_Y)\, X^T w = (X D X^T - U F^{-1} X^T)\, w = \hat{X} X^T w = \hat{X} v = 0.$$

Therefore, we have $w \in \operatorname{Ker}[X (D - D_Y)\, X^T]$ and $v = X^T w \in X^T \operatorname{Ker}[X (D - D_Y)\, X^T]$. Hence, the inclusion $\operatorname{Ker} \hat{X} \subseteq X^T \operatorname{Ker}[X (D - D_Y)\, X^T]$ holds. Conversely, assume that $v = X^T w$ with $w \in \operatorname{Ker}[X (D - D_Y)\, X^T]$. With the aid of Lemma A.1 we then get

$$\hat{X} v = (X D - U F^{-1})\, X^T w = (X D X^T - X D_Y X^T)\, w = X (D - D_Y)\, X^T w = 0,$$

i.e., the vector $v \in \operatorname{Ker} \hat{X}$. Thus, also the inclusion $X^T \operatorname{Ker}[X (D - D_Y)\, X^T] \subseteq \operatorname{Ker} \hat{X}$ holds and (A.6) is proven. Finally, it is easy to see from (A.6) that the matrix $\hat{X}$ is nonsingular, i.e., the subspace $\operatorname{Ker} \hat{X} = \{0\}$, if and only if $\operatorname{Ker}[X (D - D_Y)\, X^T] \subseteq \operatorname{Ker} X^T$. According to Remark A.3(i), the last inclusion is equivalent with equality (A.4). Finally, by the above expression of $\hat{X} = -U F^{-1} + X D$ and by Lemma A.1 we obtain $\hat{X} X^T = X (D - D_Y)\, X^T$, which implies (A.5), since $\hat{X}$ is invertible. The proof is complete. □

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